MIDTERM SOLUTIONS - MATH 2121, FALL 2018.

Name:

Student ID:

Email:

Tutorial: T1A T1B T2A T2B T3A T3B

Problem #	Max points possible	Actual score
1	10	
2	10	
3	20	
4	10	
5	10	
6	20	
Total	80	

You have **120 minutes** to complete this exam.

No books, notes, or electronic devices can be used on the test.

Draw a box around your answers or write your answers in the boxes provided.

Partial credit can be given on some problems if you show your work. Good luck!

Problem 1. (10 points)

Assume h and k are real numbers and consider the linear system

$$x_1 + 3x_2 = k$$
$$4x_1 + hx_2 = 8.$$

Determine all values of h and k such that this system has (i) zero solutions, (ii) a unique solution, or (iii) infinitely many solutions.

Solution:

Row reducing the augmented matrix of this linear system gives

$$\begin{bmatrix} 1 & 3 & k \\ 4 & h & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & k \\ 0 & h-12 & 8-4k \end{bmatrix}.$$

The matrix has a pivot in the last column if $h - 12 = 0$ and $8 - 4k \neq 0$. In this case the system has zero solutions.

The matrix has a pivot in the first two columns if $h - 12 \neq 0$. In this case the system has a unique solution.

The matrix has a pivot in only the first column if $h - 12 = 8 - 4k = 0$. In this case the system has infinitely many solutions since x_2 is a free variable.

(i) The system has zero solutions when:

$h = 12$ and $k \neq 2$

(ii) The system has a unique solution when:

$h \neq 12$ and k is any real number

(iii) The system has infinitely solutions when:

$h = 12$ and $k = 2$.

Problem 2. (10 points)

Assume A is a 3×3 matrix. The first two columns of A are pivot columns and

$$A \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

What is the reduced echelon form of A?

Solution:

Let E denote the reduced echelon form of A. Since the first two columns of A are pivot columns, E has the form

$$E = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & c \end{bmatrix}$$

for some numbers a, b, and c. Since A and E have the same nullspace, we have

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = E \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} a+3 \\ b-2 \\ c \end{bmatrix}.$$

Therefore we must have $a = -3$ and $b = 2$ and $c = 0$ so the matrix E is

$$\boxed{\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}}$$

Problem 3. (20 points) Indicate which of the following is TRUE or FALSE.

(1) If a system of linear equations has two different solutions, then it must have infinitely many solutions.
(2) If A is an $m \times n$ matrix and the equation $Ax = b$ is consistent for every b in $\mathbb{R}^m$, then A has m pivot columns.
(3) A linear system with no free variables has a unique solution.
(4) If a linear system $Ax = b$ has more than one solution, then so does the linear system $Ax = 0$.
(5) If $u, v, w \in \mathbb{R}^2$ are all nonzero, then w is a linear combination of u and v.
(6) If A, B, and C are matrices with $AB = AC$, then $B = C$.
(7) If A and B are $m \times n$ matrices, then both AB^T and A^TB are defined.
(8) If A and B are $n \times n$ matrices with $AB = BA$, and if A is invertible, then $A^{-1}B = BA^{-1}$.
(9) If two matrices are row equivalent, then they have the same column space.
(10) If two matrices are row equivalent, then they have the same null space.

Each part will be graded as follows: 0 points for a wrong answer, 1 point for no answer, 2 points for the correct answer. Explanations are not required for answers.

Solution:

(1)	TRUE	as a linear system has 0, 1, or infinitely many solutions
(2)	TRUE	if A has m pivot columns, then A has a pivot in every row
(3)	FALSE	a system with no free variables could have no solutions
(4)	TRUE	if $Ax = Ay = b$ and $x \neq y$ then $A0 = 0$ and $A(x - y) = 0$
(5)	FALSE	we could have $u = v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
(6)	FALSE	if A is the zero matrix, then $AB = AC$ even if $B \neq C$
(7)	TRUE	as A has n columns and B^T has n rows
(8)	TRUE	$A^{-1}B = A^{-1}(BA)A^{-1} = A^{-1}(AB)A^{-1} = BA^{-1}$
(9)	FALSE	consider $A = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\texttt{RREF}(A) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
(10)	TRUE	since $\texttt{RREF}(A) = EA$ where E is invertible

Problem 4. (10 points) Let A and B be matrices.

Suppose $AB = \begin{bmatrix} 9 & 8 \\ 7 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$. Find A.

Solution:

Since $9 - 8 = 1$, we have $B^{-1} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$. Therefore

$$A = ABB^{-1} = \begin{bmatrix} 9 & 8 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 27-16 & -36+24 \\ 21-12 & -28+18 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & -12 \\ 9 & -10 \end{bmatrix}.$$

Problem 5. (10 points)

(a) Give an example of a 4×3 matrix A such that the linear transformation $T(v) = Av$ is a one-to-one function $\mathbb{R}^3 \to \mathbb{R}^4$. Justify your answer.

A common example was

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

This matrix is already in reduced echelon form, and has a pivot in every column. Therefore its columns are linearly independent and $T(v) = Av$ is a one-to-one function $\mathbb{R}^3 \to \mathbb{R}^4$.

(b) Give an example of a 2×3 matrix A such that the linear transformation $T(v) = Av$ is an onto function $\mathbb{R}^3 \to \mathbb{R}^2$. Justify your answer.

A common example was

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

This matrix is already in reduced echelon form, and has a pivot in every row. Therefore $T(v) = Av$ is an onto function $\mathbb{R}^3 \to \mathbb{R}^2$.

Problem 6. (20 points) Consider the matrix

$$A = \begin{bmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -13 \end{bmatrix}.$$

Remember that

- The *column space* of A is the span of its columns.
- The *null space* of A is the set of vectors v with $Av = 0$.

(a) Compute the reduced echelon form of A.

We have

$$\begin{aligned} A &= \begin{bmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -13 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -13 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -5 & 6 \\ 3 & 4 & 8 & -13 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -5 & 6 \\ 0 & 1 & 5 & -16 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -5 & 6 \\ 0 & 0 & 0 & -10 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

The last matrix is in reduced echelon form.

(b) Find a basis for the column space of

$$A = \begin{bmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -13 \end{bmatrix}.$$

The first, second, and fourth columns of A are pivot columns, so they are a basis for the column space: $\boxed{\begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 12 \\ -13 \end{bmatrix}}$.

(c) Find a basis for the null space of

$$A = \begin{bmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -13 \end{bmatrix}.$$

We have $Ax = 0$ if and only if

$$0 = \texttt{RREF}(A)x = \begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 - 4x_3 \\ x_2 + 5x_3 \\ x_4 \end{bmatrix}.$$

The basic variables of this linear system expressed in terms of the free variables are thus

$$x_1 = 4x_3 \qquad \text{and} \qquad x_2 = -5x_3 \qquad \text{and} \qquad x_4 = 0.$$

We conclude that $x \in \mathrm{Nul} A$ if and only if

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4x_3 \\ -5x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 4 \\ -5 \\ 1 \\ 0 \end{bmatrix}.$$

The single vector $\boxed{\begin{bmatrix} 4 \\ -5 \\ 1 \\ 0 \end{bmatrix}}$ is therefore a basis for $\mathrm{Nul} A$.

(d) What are the dimensions of the column space and null space of A?

The column space has dimension 3 and the null space has dimension 1.