MIDTERM – MATH 2121, FALL 2022

Name:

Student ID:

Email:

Section: L1 L2

Tutorial: T1A T1B T1C T1D T2A T2B T2C T2D

Problem #	Points Possible	Score
1	20	
2	20	
3	20	
4	20	
Total	80	

You have **120 minutes** to complete this exam.

No books, notes, or electronic devices can be used on the test.

Draw a box around your answers or write your answers in the boxes provided.

Partial credit can be given on some problems if you show your work. Good luck!

Each problem on this exam has multiple parts.

The parts on each problem are not always weighted equally.

You should look through the whole exam when we start and work first on the parts that you find easiest.

Some reminders:

- If A is an $m \times n$ matrix then $\mathrm{Col}(A) = \{Av : v \in \mathbb{R}^n\}$.
- If A is an $m \times n$ matrix then $\mathrm{Nul}(A) = \{v \in \mathbb{R}^n : Av = 0\}$.
- If $ad - bc \neq 0$ then $\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]^{-1} = \frac{1}{ad-bc}\left[\begin{array}{rr} d & -b \\ -c & a \end{array}\right]$.

Problem 1. (20 points)

This problem has five parts.

(a) There are sixty-four 3×2 matrices whose entries are each zero or one. An example of such a matrix is $\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$. List all of the 3×2 matrices whose entries are each zero or one that are in reduced echelon form.

(b) Suppose $v_1, v_2, v_3, v_4, v_5 \in \mathbb{R}^4$ are the columns of the 4×5 matrix
$$A = \left[\begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array}\right].$$
The reduced echelon form of A is
$$\mathsf{RREF}(A) = \left[\begin{array}{ccccc} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$
What is the reduced echelon form of the matrix $B = \left[\begin{array}{ccc} v_1 & v_2 & v_3 \end{array}\right]$?

(c) Suppose $v_1, v_2, v_3, v_4, v_5 \in \mathbb{R}^4$ are the columns of the 4×5 matrix

$$A = \left[\begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array}\right].$$

The reduced echelon form of A is

$$\mathsf{RREF}(A) = \left[\begin{array}{ccccc} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

(This is the same setup as the previous part.)

What is the reduced echelon form of the matrix $C = \left[\begin{array}{ccc} v_1 & v_3 & v_5 \end{array}\right]$?

(d) Suppose $v_1, v_2, v_3, v_4, v_5 \in \mathbb{R}^4$ are the columns of the 4×5 matrix
$$A = \left[\begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \right].$$
The reduced echelon form of A is
$$\mathsf{RREF}(A) = \left[\begin{array}{ccccc} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right].$$
(This is the same setup as the previous part.)

What is the reduced echelon form of the matrix $D = \left[\begin{array}{ccc} v_3 & v_4 & v_5 \end{array} \right]$?

(e) Find the general solution to the linear system

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 1 \\ x_1 + 2x_2 + 4x_3 + 2x_4 = 0 \\ 2x_1 - 4x_3 + x_4 = 0. \end{cases}$$

Problem 2. (20 points)

Recall that the standard matrix of a linear function $f : \mathbb{R}^n \to \mathbb{R}^m$ is the matrix A such that $f(x) = Ax$ for all $x \in \mathbb{R}^n$. This problem has five parts.

(a) Find the standard matrix of the linear function $f : \mathbb{R}^4 \to \mathbb{R}^2$ defined by

$$f\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}\right) = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

(b) Find the standard matrix of the linear function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = \det \begin{bmatrix} 1 & v_1 & 2 \\ 5 & v_2 & 4 \\ 2 & v_3 & 3 \end{bmatrix}.$$

(c) Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$ are both linear functions.

Recall that $f \circ g$ is the function defined by $(f \circ g)(x) = f(g(x))$.

If f has standard matrix $\left[\begin{array}{cc} 2 & 3 \\ 3 & 5 \end{array}\right]$ and $f \circ g$ has standard matrix $\left[\begin{array}{cc} 0 & 2 \\ 3 & 0 \end{array}\right]$ then what is the standard matrix of g?

(d) Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear function with standard matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix}.$$

Determine m, n, and whether or not f is one-to-one or onto.

$m =$	$n =$

Is f one-to-one?

Justify your answer:

Is f onto?

Justify your answer:

(e) Suppose $a, b, c, d \in \mathbb{R}$ are real numbers with $ad - bc \neq 0$. There is a unique linear function $f : \mathbb{R}^2 \to \mathbb{R}^2$ with

$$f\left(\begin{bmatrix} a \\ c \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad f\left(\begin{bmatrix} b \\ d \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

What is the standard matrix of f?

Problem 3. (20 points)

Suppose A is a (not necessarily square) $m \times n$ matrix. Note that $A^T A$ is a square $n \times n$ matrix. Here are some possible properties that A could have:

(1) A is square with $m = n$.
(2) every row of A has a pivot position.
(3) every column of A has a pivot position.
(4) the equation $Ax = b$ has a solution for each $b \in \mathbb{R}^m$.
(5) the equation $Ax = b$ has a unique solution for each $b \in \mathbb{R}^m$.

(6) the equation $Ax = b$ has a unique solution for some $b \in \mathbb{R}^m$.
(7) the equation $Ax = 0$ has infinitely many solutions.
(8) the equation $Ax = 0$ has exactly one solution.
(9) the equation $Ax = 0$ does not have a solution.
(10) $\mathrm{Col}(A) = \mathbb{R}^m$.

(11) $\mathrm{Nul}(A) = \mathbb{R}^n$.
(12) $\mathrm{Col}(A) = \{0\}$.
(13) $\mathrm{Nul}(A) = \{0\}$.
(14) the columns of A are linearly independent.
(15) the span of the columns of A is $\mathbb{R}^m$.

(16) the reduced echelon form of A is an identity matrix.
(17) AB is an identity matrix for some $n \times m$ matrix B.
(18) $\mathrm{rank}(A) = \min\{m, n\}$.
(19) $\det(A^T A) \neq 0$.
(20) $(A^T A)^k$ is an identity matrix for some integer $k \geq 1$.

This problem has four parts. You do not need to provide explanations for parts (a), (b), and (c), but show your work as usual on part (d).

(a) Circle all of the properties that A must have **if A is invertible**.

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20

(b) Here are the properties listed again:

(1) A is square with $m = n$.
(2) every row of A has a pivot position.
(3) every column of A has a pivot position.
(4) the equation $Ax = b$ has a solution for each $b \in \mathbb{R}^m$.
(5) the equation $Ax = b$ has a unique solution for each $b \in \mathbb{R}^m$.

(6) the equation $Ax = b$ has a unique solution for some $b \in \mathbb{R}^m$.
(7) the equation $Ax = 0$ has infinitely many solutions.
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(10) $\mathrm{Col}(A) = \mathbb{R}^m$.

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(16) the reduced echelon form of A is an identity matrix.
(17) AB is an identity matrix for some $n \times m$ matrix B.
(18) $\mathrm{rank}(A) = \min\{m, n\}$.
(19) $\det(A^T A) \neq 0$.
(20) $(A^T A)^k$ is an identity matrix for some integer $k \geq 1$.

Circle all of the properties that **imply by themselves that** A **is invertible**.

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20

(c) Here are the properties listed again:

(1) A is square with $m = n$.
(2) every row of A has a pivot position.
(3) every column of A has a pivot position.
(4) the equation $Ax = b$ has a solution for each $b \in \mathbb{R}^m$.
(5) the equation $Ax = b$ has a unique solution for each $b \in \mathbb{R}^m$.

(6) the equation $Ax = b$ has a unique solution for some $b \in \mathbb{R}^m$.
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(12) $\mathrm{Col}(A) = \{0\}$.
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(15) the span of the columns of A is $\mathbb{R}^m$.

(16) the reduced echelon form of A is an identity matrix.
(17) AB is an identity matrix for some $n \times m$ matrix B.
(18) $\mathrm{rank}(A) = \min\{m, n\}$.
(19) $\det(A^T A) \neq 0$.
(20) $(A^T A)^k$ is an identity matrix for some integer $k \geq 1$.

Circle all of the properties that **imply by themselves that A is invertible if we also assume that A is a square matrix with $m = n$.**

	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20

(d) Compute the inverse of the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 4 \\ 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 4 & 4 & 4 & 4 \end{bmatrix}.$$

Problem 4. (20 points)

The matrix

$$A = \begin{bmatrix} 2 & 4 & 2 & 0 & 2 \\ 1 & 2 & 0 & 1 & 3 \\ 2 & 4 & 2 & 0 & 2 \\ 1 & 2 & 2 & -1 & 7 \end{bmatrix}$$

has reduced echelon form

$$\mathsf{RREF}(A) = \begin{bmatrix} 1 & 2 & 0 & 1 & 3 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This question has four parts.

(a) Find a basis for the column space of A.

(b) Find a basis for the null space of A.

(c) Find all pairs of real numbers (x, y) such that $\begin{bmatrix} 2 \\ x \\ 2 \\ y \end{bmatrix}$ is in $\mathrm{Col}(A)$.

(d) Find all pairs of real numbers (x, y) such that $\begin{bmatrix} x \\ 0 \\ y \\ 1 \\ 1 \end{bmatrix}$ is in $\mathrm{Nul}(A)$.