Instructions: Choose **4 problems** and write down detailed solutions, showing all necessary work. You can earn up to **4 extra credit points** by correctly solving additional problems.[1]

Some of the problems are more challenging than others, and there is no need to solve all of them. Problems that would not make reasonable exam questions (either because of difficulty, being open-ended, or requiring external resources) are marked with a star. These problems may still offer useful practice with the core concepts in the course.

You are free to discuss problems with other students and to consult whatever resources you want, but you must write up your own solutions. If your solutions appear to be copied from somewhere else, you will automatically receive zero credit. **Please handwrite your answers and show all steps in your calculations**, as you would on an exam.

To get full credit for the offline homework, you just need to make a good-faith attempt on the required problems. The bar for receiving extra credit points is higher.

Show all steps and provide justification for all answers.

In these exercises, a matrix A is *diagonalizable over* $\mathbb{R}$ if $A = PDP^{-1}$ for an invertible matrix P with all real entries and a diagonal matrix D with all real entries. A matrix A is *diagonalizable over* $\mathbb{C}$ if $A = PDP^{-1}$ for an invertible matrix P, possibly with complex entries, and a diagonal matrix D, possibly with complex entries. If we just say "diagonalizable" then we mean "diagonalizable over $\mathbb{C}$".

1. Let $A = \begin{bmatrix} .4 & -.3 \\ .4 & 1.2 \end{bmatrix}$. Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

   Then compute $\lim_{n\to\infty} A^n$.

2. Determine if $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & -1 & -2 \\ 3 & 3 & 4 \end{bmatrix}$ is diagonalizable. If A is diagonalizable, then find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

*3. Consider the integer sequence P_n defined by the recurrence $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$, where $P_0 = 0$ and $P_1 = 1$. Check that

$$\begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P_n & P_{n-1} \\ P_{n-1} & P_{n-2} \end{bmatrix}$$

for all integers $n \geq 2$. Explain why this implies that

$$\begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n$$

for all integers $n \geq 0$, and use this to derive an exact formula for P_n similar to the formula we found in class for the Fibonacci numbers.

4. Suppose A is a 3×3 matrix such that

$$A \begin{bmatrix} 13 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}, \quad A \begin{bmatrix} 11 \\ 12 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -4 \end{bmatrix}, \quad A \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$

   Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

   Then determine if $\lim_{n\to\infty} A^n$ exists and compute its value if it does.

[1] There will be ∼10 weeks of assignments, each with ∼10 practice problems, so you can earn up to ∼40 equally weighted extra credit points. The maximum amount of extra credit you can earn is 5% of your total grade for the semester.

5. Give an example of each of the following, or explain why no such example exists:
   (a) A diagonalizable matrix that is not diagonal.
   (b) A diagonal matrix that is not diagonalizable.
   (c) A diagonalizable matrix that is not triangular.
   (d) A triangular matrix that is not diagonalizable.
   (e) An invertible matrix that is not triangular.
   (f) A triangular matrix that is not invertible.
   (g) An invertible matrix that is not diagonalizable.
   (h) A diagonalizable matrix that is not invertible.

   All matrices in this problem are assumed to be square with all real entries, and "diagonalizable" means "diagonalizable over the complex numbers."

6. Suppose A is a 3×3 matrix (with all real entries, as usual) that has exactly two distinct (complex) eigenvalues $\lambda \neq \mu$, and that has the following vectors as eigenvectors (each with eigenvalue λ or μ):

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

   (a) Explain why you must have $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$.
   (b) What conditions must λ and μ satisfy for A to be diagonalizable?

   What conditions must λ and μ satisfy for A to be non-invertible?

   (c) Fix a choice of λ and μ satisfying the conditions in A. Then find **two examples** of matrices A with the given properties (that is, having only λ and μ as eigenvalues, having the given vectors as eigenvectors, and which are diagonalizable but not invertible) **which are not similar**.

∗7. If $\theta \in \mathbb{R}$ then $e^{i\theta} \in \mathbb{C}$ is **defined** to be the complex number $\cos\theta + i \sin\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

   If $\theta = 0$ then $e^{i\theta} = 1$. Also: $e^{i\theta_1} = e^{i\theta_2}$ for $\theta_1, \theta_2 \in \mathbb{R}$ if and only if $\theta_1 - \theta_2$ is an integer times 2π.

   (a) Explain why in this notation we have $e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$ for all $\theta_1, \theta_2 \in \mathbb{R}$.

   Then use this notation to find all (complex) solutions to the equation $x^n = 1$.

   Then also find all (complex) solutions to the equation $x^n = -1$.

   (Here n is an arbitrary positive integer, as usual.)

   (b) Find the characteristic polynomial of the $n \times n$ *cyclic permutation matrix*

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} = \begin{bmatrix} e_2 & e_3 & e_4 & \dots & e_n & e_1 \end{bmatrix}.$$

   Then use (a) to deduce that A is diagonalizable (over the complex numbers).

   (c) Explain why if B is a diagonalizable $m \times m$ matrix and C is a diagonalizable $n \times n$ matrix then the $(m+n) \times (m+n)$ matrix $\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$ is also diagonalizable.

   We call the last matrix the *direct sum* of B and C.

Any permutation matrix is similar to a direct sum of cyclic permutation matrices (in fact, if Q is a permutation matrix then we can always find a permutation matrix P such that PQP^{-1} is such a direct sum), so this exercise show that **every permutation matrix is diagonalizable** over $\mathbb{C}$.

8. Suppose $a, b, c, d \in \mathbb{R}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

   (a) When does A have two distinct real eigenvalues (so A is diagonalizable over $\mathbb{R}$)?

   (b) When does A have two distinct complex non-real eigenvalues (so A is diagonalizable over $\mathbb{C}$)?

   (c) When does A have exactly one complex eigenvalue λ? In this case what is λ?

   (d) When is A **not** diagonalizable over $\mathbb{C}$?

   Your answer to each part should consist of precise conditions in terms of a, b, c, d.

$*$9. An $n \times n$ matrix A is *nilpotent* if $A^k = 0$ for some integer $k > 0$.

   For example, the zero $n \times n$ matrix is nilpotent. Also, any strictly upper triangular or strictly lower triangular $n \times n$ matrix (meaning triangular with all zeros on the diagonal) is nilpotent.

   First, give an example of a nilpotent matrix that is not strictly upper or lower triangular.

   Next, explain why if A is an $n \times n$ nilpotent matrix **that is nonzero**, then (a) the only eigenvalue of A is zero, but (b) the null space of A is not n-dimensional, and therefore (c) A is not diagonalizable.

$**$10. The hour hand of a faulty 12-hour clock points to the correct hour at time $n = 0$. Every sixty minutes the hour hand moves forward one hour with probability $\frac{1}{2}$, does not move at all with probability $\frac{1}{4}$, or moves forward two hours with probability $\frac{1}{4}$. Find an exact formula for the probability that the hour hand points to the correct time after n hours.

$*$11. You have two large cups.

   At the start, cup #1 contains 1 liter of milk and one cup #2 contains 1 liter of coffee.

   You have a third, smaller cup #3, whose volume is r liters for some $0 < r < 1$.

   Both of the larger cups can contain $1 + r$ liters without spilling. So consider the following process:

   At each step, you pour out cup #1 to fill cup #3, and then pour all of cup #3 into cup #2. After thoroughly mixing cup #2, you pour out cup #2 to fill cup #3, and finally pour out cup #3 back into cup #1 and thoroughly mix its contents.

   Let a_n be the amount of milk in cup #1 after step n.

   Let b_n be the amount of coffee in cup #2 after step n.

   For example, $a_0 = b_0 = 1$ and $b_1 = \frac{1}{1+r}$ and $a_1 = (1 - r) + r \cdot \frac{r}{1+r} = \frac{1}{1+r}$.

   (a) Using symmetry or another reason, explain why $a_n = b_n$ for all $n = 0, 1, 2, \ldots$.

   (b) Find an exact formula for $a_n = b_n$ as a function of r and n.

   (c) If $r = 0.001$ (one milliliter) then what is the smallest n with $a_n < 0.75$?

12. Show that if D is an $n \times n$ diagonal matrix, then D is similar to any $n \times n$ diagonal matrix E formed by rearranging its diagonal entries. For example, the diagonal matrices

$$\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \lambda_n & & & \\ & \ddots & & \\ & & \lambda_2 & \\ & & & \lambda_1 \end{bmatrix}$$

   are similar.