Instructions: Choose **4 problems** and write down detailed solutions, showing all necessary work. You can earn up to **4 extra credit points** by correctly solving additional problems.[1]

Some of the problems are more challenging than others, and there is no need to solve all of them. Problems that would not make reasonable exam questions (either because of difficulty, being open-ended, or requiring external resources) are marked with a star. These problems may still offer useful practice with the core concepts in the course.

You are free to discuss problems with other students and to consult whatever resources you want, but you must write up your own solutions. If your solutions appear to be copied from somewhere else, you will automatically receive zero credit. **Please handwrite your answers and show all steps in your calculations**, as you would on an exam.

To get full credit for the offline homework, you just need to make a good-faith attempt on the required problems. The bar for receiving extra credit points is higher.

Show all steps and provide justification for all answers.

1. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \\ -1 & 1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 6 \\ 5 \\ 5 \end{bmatrix}$.

   Find the orthogonal projection of b onto the orthogonal complement of the column space of A.

   Show all the steps in your derivation to receive credit.

2. Find all least-squares solutions to the linear equation $Ax = b$ where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}.$$

   Show all the steps in your derivation to receive credit.

3. Suppose $A = \begin{bmatrix} 1 & 3 \\ 0 & -1 \\ 2 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 1 \\ h \end{bmatrix}$.

   (a) For which values of $h \in \mathbb{R}$ does $Ax = b$ have an exact solution?

   For such h, find all exact solutions to $Ax = b$.

   (b) For which values of $h \in \mathbb{R}$ does $Ax = b$ have a least-squares solution?

   For such h, find all least-squares solutions to $Ax = b$.

   Show all the steps in your derivation to receive credit.

4. Suppose $b \in \mathbb{R}^n$ is a list of numeric data.

   There exists a matrix A such that the *mean* of b is the unique least-squares solution $\widehat{x}$ to $Ax = b$ and the *(uncorrected sample) standard deviation* of b is $\frac{1}{\sqrt{n}}\|A\widehat{x} - b\|$.

   What is A?

   Explain your answer (which should depend on n but not on b).

[1] There will be ∼10 weeks of assignments, each with ∼10 practice problems, so you can earn up to ∼40 equally weighted extra credit points. The maximum amount of extra credit you can earn is 5% of your total grade for the semester.

5. Suppose $A = \begin{bmatrix} 3 & -6 & 0 \\ -6 & 0 & 6 \\ 0 & 6 & -3 \end{bmatrix}$.

   Find an orthogonal matrix U and a diagonal matrix D such that $A = UDU^\top$.

   Show all the steps in your derivation to receive credit.

6. Suppose $A = A^\top$ is a symmetric $n \times n$ matrix with all real entries.

   (a) Explain why $v^\top Av > 0$ for all nonzero $v \in \mathbb{R}^n$ if A has all positive eigenvalues.

   (b) Explain why $v^\top Av = 0$ for some nonzero $v \in \mathbb{R}^n$ if A has zero as an eigenvalue.

   (c) Explain why $v^\top Av < 0$ for some $v \in \mathbb{R}^n$ if A has a negative eigenvalue.

   This shows that $v^\top Av > 0$ for all nonzero $v \in \mathbb{R}^n$ if and only if A has all positive eigenvalues.

   A symmetric matrix with this property is called *positive definite*.

   This also shows that $v^\top Av \geq 0$ for all $v \in \mathbb{R}^n$ if and only if A has all nonnegative eigenvalues.

   A symmetric matrix with this property is called *positive semidefinite*.

7. Suppose $A = A^\top$ is a symmetric $n \times n$ matrix with all real entries.

   Explain why $A = B^\top B$ for some $n \times n$ matrix B (possibly with complex entries).

   Compute a value for B when $A = \begin{bmatrix} 3 & -6 & 0 \\ -6 & 0 & 6 \\ 0 & 6 & -3 \end{bmatrix}$.

8. Suppose $A = A^\top$ is a positive semidefinite symmetric $n \times n$ matrix with all real entries.

   Explain how to find a positive semidefinite symmetric $n \times n$ matrix B with all real entries such that $A = B^2$. Compute B when $A = \begin{bmatrix} 40 & -28 & -26 \\ -28 & 52 & 2 \\ -26 & 2 & 25 \end{bmatrix}$.

*9. Find the values of $x, y \in \mathbb{R}$ that minimize the distance between the vectors $\begin{bmatrix} x \\ x \\ x \end{bmatrix}$ and $\begin{bmatrix} y \\ 3y \\ -1 \end{bmatrix}$.

   Show all the steps in your derivation to receive credit.

*10. Suppose $a_1 < a_2 < a_3$ are real numbers and $b_1, b_2, b_3 \in \mathbb{R}$.

   Explain why the points $(x, y) = (a_1, b_1), (a_2, b_2), (a_3, b_3)$ are all on the same line if and only if

$$a_1(b_3 - b_2) + a_2(b_1 - b_3) + a_3(b_2 - b_1) = 0.$$

   (One approach: consider the line of best fit through the three points.)