SCATTERING AND FIELD ENHANCEMENT OF A PERFECT CONDUCTING NARROW SLIT *

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Abstract. This paper is concerned with the scattering and field enhancement of a narrow slit perforated in a slab of perfect conductor. We demonstrate that the enhancement of the electromagnetic field for such a configuration can be induced by either Fabry-Perot type scattering resonances or certain non-resonant effect in the quasi-static regime. We derive the asymptotic expansions of Fabry-Perot type resonances and quantitatively analyze the field enhancement at the resonant frequencies, for which both the enhancement order and the shapes of resonant modes are precisely characterized. The field enhancement at non-resonant frequencies in the quasi-static regime is also investigated. It is shown that the fast transition of the magnetic field in the slit induces strong electric field enhancement.

Key words. Electromagnetic field enhancement, narrow slit, subwavelength structures, Fabry-Perot resonance, Helmholtz equation, asymptotic analysis

AMS subject classifications. 35C20, 35Q60, 35P30.

1. Introduction. Electromagnetic scattering by subwavelength metallic structures such as apertures and holes has drawn much attention in recent years. See, for instance, [6, 9, 12, 13, 16, 18, 23, 24, 25, 26] and the references therein. The main motivation for the development of such tiny structures is due to their abilities to generate extraordinary optical transmission and strongly enhanced local electromagnetic fields, which lead to potentially important applications in biological and chemical sensing, near-field spectroscopy, and design of novel optical devices. The mechanism of the enhancement, however, varies from case to case. It could be excited by surface plasmonic resonance as claimed in [12, 13], or by non-plasmonic resonances as investigated in [25, 26], or even without the resonant effect (cf. [18, 19]). In more complicated scenarios, surface plasmonic resonance may couple with other mechanisms to yield strong enhancement, and there are still debates over which mechanism is dominant [13]. In this paper, we aim to present a quantitative analysis of the field enhancement for the electromagnetic scattering by a narrow slit and give a complete picture for the mechanism of such enhancement. The case of a perfect conductor is investigated, which excludes the existence of surface plasmonic resonance. The configuration with periodic narrow slits as well as with surface plasmonic resonance effects will be reported in forthcoming papers.

The slit is perforated in a slab of perfect conductor and the geometry of its cross section is depicted in Figure 1.1. The slab occupies the domain $\{(x_1, x_2) \mid 0 < x_2 < \ell\}$ on the x_1x_2 plane, and the slit, which is invariant along the x_3 direction, has a rectangular cross section $S_\varepsilon := \{(x_1, x_2) \mid 0 < x_1 < \varepsilon, 0 < x_2 < \ell\}$. We are interested in the case when the width of slit is much smaller compared to the thickness of the slab ℓ and the wavelength of the incident field λ, i.e., $\varepsilon \ll \ell$ and $\varepsilon \ll \lambda$. For clarity of exposition, we shall assume $\ell = 1$ in all technical derivations throughout the paper. The case of $\ell \neq 1$ follows by a simple scaling argument, and the corresponding

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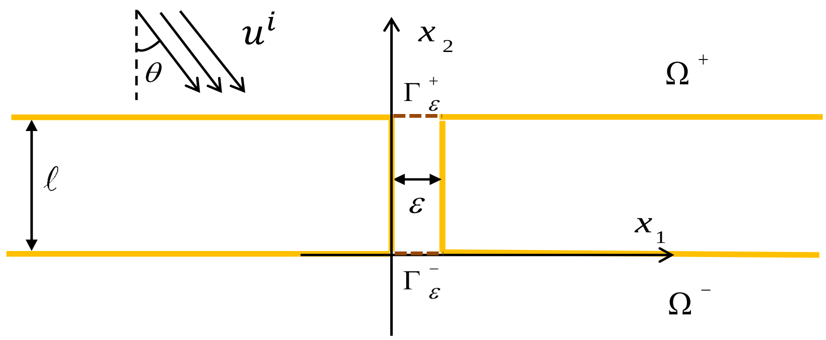

FIG. 1.1. *Geometry of the problem. The slit S_ε has a rectangular shape of length ℓ and width ε respectively. The domains above and below the perfect conductor slab are denoted as Ω_+ and Ω_- respectively, and the domain exterior to the perfect conductor is denoted as Ω_ε, which consists of S_ε, Ω_+, and Ω_-.*

enhancement theory will also be presented. Let us denote by Γ_ε^+, Γ_ε^- the upper and lower aperture of the slit, and Ω^+, Ω^- the semi-infinite domain above and below the slab respectively. We also denote by Ω_ε the domain exterior to the perfect conductor, i.e. $\Omega_\varepsilon = \Omega^+ \cup \Omega^- \cup S_\varepsilon$, and ν the unit outward normal pointing to the exterior domain Ω_ε.

We consider the scattering when a polarized time-harmonic electromagnetic wave impinges upon the perfect conductor. The transverse magnetic (TM) case is considered here by assuming that the incident magnetic field $H^i = (0, 0, u^i)$, where $u^i = e^{ik(d_1 x_1 - d_2(x_2-\ell))}$ is a plane wave, k is the wavenumber and $d = (d_1, -d_2)$ is the direction unit vector of incidence with $d_2 > 0$. If there is no slit, the total field above the slab, i.e. in the region Ω_+, is the superposition of the incident field u^i and the reflected field $u^r = e^{ik(d_1 x_1 + d_2(x_2-\ell))}$. The total field below the slab, i.e. in the region Ω^-, is equal to zero. In the presence of the slit S_ε, the total field, denoted by u_ε, consists of three parts in the upper domain Ω^+: the incident field u^i, the reflected field u^r, and the scattered field u_ε^s radiating from the aperture Γ_ε^+. In the lower domain Ω^-, the total field only consists of the scattered field radiating from the aperture Γ_ε^-. For a perfect conductor, we have the boundary condition $\dfrac{\partial u_\varepsilon}{\partial \nu} = 0$ on $\partial\Omega_\varepsilon$. In addition, at infinity the scattered field u_ε^s satisfies the Sommerfeld radiation condition [11, 20]. Therefore, the scattering problem is modeled by the following equations:

$$
\left\{
\begin{aligned}
&\Delta u_\varepsilon + k^2 u_\varepsilon = 0 \qquad \text{in } \Omega_\varepsilon, \\
&\frac{\partial u_\varepsilon}{\partial \nu} = 0 \quad \text{on } \partial\Omega_\varepsilon. \\
&u_\varepsilon = u^i + u^r + u_\varepsilon^s, \quad \text{in } \Omega_\varepsilon^+, \\
&u_\varepsilon = u_\varepsilon^s, \quad \text{in } \Omega_\varepsilon^-, \\
&\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u_\varepsilon^s}{\partial r} - iku_\varepsilon^s\right) = 0, \qquad r = |x|\,.
\end{aligned}
\right.
\tag{1.1}
$$

It is known that (see, for instance [19]) the above scattering problem attains a unique solution for k with $\mathrm{Im}k \geq 0$. Using analytic continuation, the solution also exists and is unique for all complex wavenumbers except for a countable number of points, which are poles of the resolvent associated with the scattering problem (1.1). These poles are called the resonances (or scattering resonances) of the scattering

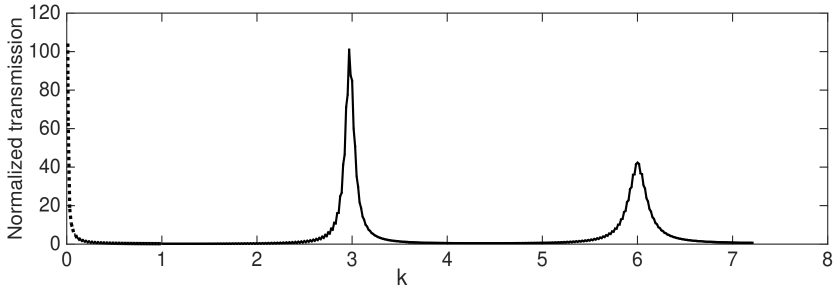

FIG. 1.2. *Normalized transmission with $\varepsilon = 0.02$ and $\ell = 1$. The normalized transmission is defined by P/P_{inc}, where P and P_{inc} denotes the transmission power and the incident power over the lower slit aperture respectively.*

problem, and the associated nontrivial solutions are called resonance modes (or quasi-normal modes). If the frequency of the incident wave is close the real part of the resonance (resonance frequency), an enhancement of scattering is expected if the imaginary part of the resonance is small. This is the mechanism of resonant scattering.

In this paper, we point out that the electromagnetic field enhancement for a narrow slit can be attributed to either resonance or certain non-resonant effect. This is illustrated in Figure 1.2, where the normalized transmission through the slit is shown for a configuration with $\varepsilon = 0.02$ and $\ell = 1$. The transmission peaks correspond to the enhancement of electromagnetic fields at specific frequencies. As to be shown in the paper, the peaks in the solid line correspond to the enhancement at resonant frequencies, or referred to as Fabry-Perot type resonances that are reported in [25]. On the other hand, no resonance exists when the frequency approaches zero, and the extraordinary transmission in the quasi-static regime (dotted line in Figure 1.2) is induced by certain non-resonant effect explained in Section 6. The goal of the paper is to (i) prove rigorously the existence of Fabry-Perot type resonances and derive the asymptotic expansions for those resonances; (ii) analyze quantitatively the field enhancement at both resonant frequencies and non-resonant frequencies in the quasi-static regime. In particular, it is shown that enhancement with an order of $O(1/\varepsilon)$ occurs at the resonant frequencies, while the enhancement is of order $O\left(1/(k\ell)\right)$ at non-resonant frequencies in the quasi-static regime. We also characterize the shapes of enhanced wave modes for both cases.

It should be mentioned that mathematical studies of the slit scattering problem have also been carried out previously in [10, 20, 21, 22], where matched asymptotic expansion techniques are applied to construct the solution of the scattering problem, including the resonance case. The field enhancement at low frequencies has been investigated by the authors in [19], assuming that the wavelength is much larger than the thickness of the slab. We also refer to a closely related problem of scattering by sub-wavelength cavities in [7, 8], where the layer potential techniques and Gohberg-Sigal theory are applied to study the resonances. A nice introduction to these techniques is given in [3]. The techniques adopted in this paper for the analysis of resonances share the same spirit as the ones used in [7, 8]. However, we avoid the operator version of residue theorem and Gohberg-Sigal theory by reducing the problem to the analysis of ordinary analytic functions. Moreover, here we also analyze quantitatively the scattering and field enhancement at both resonant and non-resonant frequencies. Finally, we refer the readers to [4, 5] for the study of closely related Helmholtz resonators and resonances in bubbly media.

The rest of the paper is organized as follows. In Section 2, we reformulate the scattering problem as equivalent boundary-integral equations. The asymptotic expansions of the boundary-integral operators are presented in Section 3. In Section 4, we prove rigorously the existence of Fabry-Perot resonances and derive their asymptotic expansions, followed by quantitatively analysis of the scattering and field enhancement at resonant frequencies in Section 5. The field enhancement in the non-resonant quasi-static regime is studied in Section 6. We end the paper with some concluding remarks in Section 7.

2. Boundary-integral formulation of the scattering problem. As mentioned in Section 1, here and henceforth, we will assume that $\ell = 1$ in all technical derivations. First, let us introduce two Green's functions, $g^e(k;x,y)$ and $g^i_\varepsilon(k;x,y)$, for the Helmholtz equation with Neumann boundary condition in the domains $\Omega^\pm$ and S_ε respectively. They satisfy the following equations

$$\begin{aligned}
\Delta g^e(k;x,y) + k^2 g^e(k;x,y) &= \delta(x-y), \quad x,y \in \Omega^\pm, \\
\Delta g^i_\varepsilon(k;x,y) + k^2 g^i_\varepsilon(k;x,y) &= \delta(x-y), \quad x,y \in S_\varepsilon.
\end{aligned}$$

In addition, $\dfrac{\partial g^e(k;x,y)}{\partial \nu_y} = 0$ for $y_2 = 1$ and $x_2 > 1$ or for $y_2 = 0$ and $x_2 < 0$, and $\dfrac{\partial g^i(k;x,y)}{\partial \nu_y} = 0$ on ∂S_ε. One can check that

$$g^e(k;x,y) = -\frac{i}{4}\left(H_0^{(1)}(k|x-y|) + H_0^{(1)}(k|x'-y|)\right),$$

where $H_0^{(1)}$ is the first-kind Hankel function of order 0, and

$$x' = \begin{cases} (x_1, 2-x_2) & \text{if } x,y \in \Omega^+, \\ (x_1, -x_2) & \text{if } x,y \in \Omega^-. \end{cases}$$

It is clear that $g_e(k;x,y) = g_e(k;y,x)$. The Hankel function can be continued analytically to the entire complex plane without the negative real axis (cf. [1]), thus $g^e(k;x,y)$ is analytic on the complex k-plane without the negative real axis. The Green function in the domain S_ε takes the following form:

$$g^i_\varepsilon(k;x,y) = \sum_{m,n=0}^{\infty} c_{mn}\phi_{mn}(x)\phi_{mn}(y),$$

where $c_{mn} = \dfrac{1}{k^2 - (m\pi/\varepsilon)^2 - (n\pi)^2}$, $\phi_{mn}(x) = \sqrt{\dfrac{\alpha_{mn}}{\varepsilon}}\cos\left(\dfrac{m\pi x_1}{\varepsilon}\right)\cos(n\pi x_2)$, and

$$\alpha_{mn} = \begin{cases} 1 & m = n = 0, \\ 2 & m = 0, n \geq 1 \quad \text{or} \quad n = 0, m \geq 1, \\ 4 & m \geq 1, n \geq 1. \end{cases}$$

Note that the above expansion of $g^i_\varepsilon(k;x,y)$ is well defined on the whole complex k-plane except for $k = \pm\sqrt{(m\pi/\varepsilon)^2 + (n\pi)^2}$, the eigenvalues of the Laplacian in S_e. We shall take a limiting procedure for the evaluation of $g^i_\varepsilon(k;x,y)$ when k are those eigenvalues. Similar procedures are also used in subsequent analysis.

Noting that $\dfrac{\partial u^i}{\partial \nu} + \dfrac{\partial u^r}{\partial \nu} = 0$ on $\{x_2 = 1\}$, it follows that the scattered field satisfies $\dfrac{\partial u^s_\varepsilon}{\partial \nu} = 0$ on $\{x_2 = 1\}\backslash\Gamma^+_\varepsilon$, and since u^s_ε is radiating, from the Green's identity one obtains an integral equation for u^s_ε involving an integral only over Γ^+_ε. Therefore, the total field is

$$u_\varepsilon(x) = \int_{\Gamma^+_\varepsilon} g^e(k;x,y)\frac{\partial u_\varepsilon(y)}{\partial \nu} ds_y + u^i(x) + u^r(x), \quad x \in \Omega^+.$$

By the continuity of the single-layer potential (cf. [15]), we have

$$u_\varepsilon(x) = \int_{\Gamma^+_\varepsilon} \left(-\frac{i}{2}\right) H^{(1)}_0(k|x-y|)\frac{\partial u_\varepsilon(y)}{\partial \nu} ds_y + u^i(x) + u^r(x) \quad \text{for } x \in \Gamma^+_\varepsilon. \tag{2.1}$$

Similarly,

$$u_\varepsilon(x) = \int_{\Gamma^-_\varepsilon} \left(-\frac{i}{2}\right) H^{(1)}_0(k|x-y|)\frac{\partial u_\varepsilon(y)}{\partial \nu} ds_y \quad \text{for } x \in \Gamma^-_\varepsilon. \tag{2.2}$$

The solution inside the slit can be expressed as

$$u_\varepsilon(x) = -\int_{\Gamma^+_\varepsilon \cup \Gamma^-_\varepsilon} g^i_\varepsilon(k;x,y)\frac{\partial u_\varepsilon(y)}{\partial \nu} ds_y \quad \text{for } x \in S_\varepsilon.$$

Again by the continuity of the single-layer potential we have

$$u_\varepsilon(x) = -\int_{\Gamma^+_\varepsilon \cup \Gamma^-_\varepsilon} g^i_\varepsilon(k;x,y)\frac{\partial u_\varepsilon(y)}{\partial \nu} ds_y \quad \text{for } x \in \Gamma^+_\varepsilon \cup \Gamma^-_\varepsilon. \tag{2.3}$$

Therefore, by imposing the continuity of the solution along the gap apertures, we obtain the following system of boundary-integral equations for $\left.\dfrac{\partial u_\varepsilon}{\partial \nu}\right|_{\Gamma^+_\varepsilon \cup \Gamma^-_\varepsilon}$:

$$\begin{cases} \displaystyle\int_{\Gamma^+_\varepsilon} \left(-\frac{i}{2}\right) H^{(1)}_0(k|x-y|)\frac{\partial u_\varepsilon(y)}{\partial \nu} ds_y + \int_{\Gamma^+_\varepsilon \cup \Gamma^-_\varepsilon} g^i_\varepsilon(k;x,y)\frac{\partial u_\varepsilon(y)}{\partial \nu} ds_y + u^i + u^r = 0 \quad \text{on } \Gamma^+_\varepsilon, \\ \displaystyle\int_{\Gamma^-_\varepsilon} \left(-\frac{i}{2}\right) H^{(1)}_0(k|x-y|)\frac{\partial u_\varepsilon(y)}{\partial \nu} ds_y + \int_{\Gamma^+_\varepsilon \cup \Gamma^-_\varepsilon} g^i_\varepsilon(k;x,y)\frac{\partial u_\varepsilon(y)}{\partial \nu} ds_y = 0 \quad \text{on } \Gamma^-_\varepsilon. \end{cases} \tag{2.4}$$

To sum up, we have the following proposition.

PROPOSITION 2.1. *The scattering problems* (1.1) *is equivalent to the system of boundary-integral equations* (2.4).

It is clear that

$$\left.\frac{\partial u_\varepsilon}{\partial \nu}\right|_{\Gamma^+_\varepsilon} = \frac{\partial u_\varepsilon}{\partial y_2}(y_1, 1), \quad \left.\frac{\partial u_\varepsilon}{\partial \nu}\right|_{\Gamma^-_\varepsilon} = -\frac{\partial u_\varepsilon}{\partial y_2}(y_1, 0), \quad (u^i + u^r)|_{\Gamma^+_\varepsilon} = 2e^{ikd_1x_1}.$$

Note that the above functions are defined over narrow intervals with size $\varepsilon \ll 1$. To facilitate the analysis, we shall rescale the functions by introducing $X = x_1/\varepsilon$ and

$Y = y_1/\varepsilon$. Let us define the following quantities:

$$\varphi_1(Y) := -\frac{\partial u_\varepsilon}{\partial y_2}(\varepsilon Y, 1);$$
$$\varphi_2(Y) := \frac{\partial u_\varepsilon}{\partial y_2}(\varepsilon Y, 0);$$
$$f(X) := (u^i + u^r)(\varepsilon X, 1) = 2e^{ikd_1\varepsilon X};$$
$$G^e_\varepsilon(X, Y) := \left(-\frac{i}{2}\right) H_0^{(1)}(\varepsilon k|X - Y|);$$
$$G^i_\varepsilon(X, Y) := g^i_\varepsilon(k; \varepsilon X, 1; \varepsilon Y, 1) = g^i_\varepsilon(k; \varepsilon X, 0; \varepsilon Y, 0) = \sum_{m,n=0}^{\infty} \frac{c_{mn}\alpha_{mn}}{\varepsilon} \cos(m\pi X) \cos(m\pi Y);$$
$$\tilde{G}^i_\varepsilon(X, Y) := g^i_\varepsilon(k; \varepsilon X, 1; \varepsilon Y, 0) = g^i_\varepsilon(k; \varepsilon X, 0; \varepsilon Y, 1) = \sum_{m,n=0}^{\infty} \frac{(-1)^n c_{mn}\alpha_{mn}}{\varepsilon} \cos(m\pi X) \cos(m\pi Y);$$

We also define three boundary-integral operators:

$$(T^e\varphi)(X) = \int_0^1 G^e_\varepsilon(X, Y)\varphi(Y)dY \quad X \in (0, 1); \tag{2.4}$$
$$(T^i\varphi)(X) = \int_0^1 G^i_\varepsilon(X, Y)\varphi(Y)dY \quad X \in (0, 1); \tag{2.5}$$
$$(\tilde{T}^i\varphi)(X) = \int_0^1 \tilde{G}^i_\varepsilon(X, Y)\varphi(Y)dY \quad X \in (0, 1). \tag{2.6}$$

By a change of variable $x_1 = \varepsilon X$ and $y_1 = \varepsilon Y$ in (2.4), the following proposition follows.

Proposition 2.2. *The system of integral equations* (2.4) *is equivalent to*

$$\begin{bmatrix} T^e + T^i & \tilde{T}^i \\ \tilde{T}^i & T^e + T^i \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} f/\varepsilon \\ 0 \end{bmatrix}. \tag{2.7}$$

3. Asymptotic analysis of the boundary-integral operators.

3.1. Preliminaries. Let $s \in \mathbf{R}$, we denote by $H^s(\mathbf{R})$ the standard fractional Sobolev space with the norm

$$\|u\|^2_{H^s(\mathbf{R})} = \int_{\mathbf{R}} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi,$$

where $\hat{u}$ is the Fourier transform of u.

Let I be a bounded open interval in $\mathbf{R}$ and define

$$H^s(I) := \{u = U|_I \bigm| U \in H^s(\mathbf{R})\}.$$

Then $H^s(I)$ is a Hilbert space with the norm

$$\|u\|_{H^s(I)} = \inf\{\|U\|_{H^s(\mathbf{R})} \bigm| U \in H^s(\mathbf{R}) \text{ and } U|_I = u\}.$$

We also define

$$\tilde{H}^s(I) := \{u = U|_I \bigm| U \in H^s(\mathbf{R}) \text{ and } supp\, U \subset \bar{I}\}.$$

One can show that (see, [2]) the space $\tilde{H}^s(I)$ is the dual of $H^{-s}(I)$ and the norm for $\tilde{H}^s(I)$ can be defined via the duality. As such $\tilde{H}^s(I)$ is also a Hilbert space. We refer to [2] for more details about the fractional Sobolev spaces.

For simplicity, we denote $V_1 = \tilde{H}^{-\frac{1}{2}}(0,1)$ and $V_2 = H^{\frac{1}{2}}(0,1)$. The duality between V_1 and V_2 will be denoted by $\langle u, v\rangle$ for any $u \in V_1$, $v \in V_2$.

3.2. Asymptotic expansions. For clarity, first let us introduce several notations below.

$$\beta_1(k,\varepsilon) = \frac{1}{\pi}(\ln k + \gamma_0) + \frac{1}{\pi}\ln\varepsilon, \tag{3.1}$$

$$\beta_2(k,\varepsilon) = \frac{\cot k}{k\varepsilon} + \frac{2\ln 2}{\pi}, \tag{3.2}$$

$$\beta(k,\varepsilon) = \beta_1(k,\varepsilon) + \beta_2(k,\varepsilon) = \frac{\cot k}{k\varepsilon} + \frac{1}{\pi}(2\ln 2 + \ln k + \gamma_0) + \frac{1}{\pi}\ln\varepsilon, \tag{3.3}$$

$$\tilde{\beta}(k,\varepsilon) = \frac{1}{(k\sin k)\varepsilon}, \tag{3.4}$$

$$\begin{aligned}\kappa(X,Y) = \frac{1}{\pi}\Bigg[&\ln\left(\left|\sin\left(\frac{\pi(X-Y)}{2}\right)\right|\right) + \ln\left(\left|\sin\left(\frac{\pi(X+Y)}{2}\right)\right|\right) \\ &+ \ln|X-Y| \Bigg].\end{aligned} \tag{3.5}$$

Here $\gamma_0 = c_0 - \ln 2 - i\pi/2$, and c_0 is the Euler constant. We have the following asymptotic expansions for the kernels G^i_ε, G^e_ε and $\tilde{G}^i_\varepsilon$.

Lemma 3.1. *If* $|k\varepsilon| \ll 1$, *then*

$$G^e_\varepsilon(X,Y) = \beta_1(k,\varepsilon) + \frac{1}{\pi}\ln|X-Y| + r^\varepsilon_1(X,Y), \tag{3.6}$$

$$\begin{aligned}G^i_\varepsilon(X,Y) = \beta_2(k,\varepsilon) + \frac{1}{\pi}\Bigg[&\ln\left(\left|\sin\left(\frac{\pi(X+Y)}{2}\right)\right|\right) \\ &+ \ln\left(\left|\sin\left(\frac{\pi(X-Y)}{2}\right)\right|\right)\Bigg] + r^\varepsilon_2(X,Y),\end{aligned} \tag{3.7}$$

$$\tilde{G}^i_\varepsilon(X,Y) = \tilde{\beta}(k,\varepsilon) + \tilde{\kappa}_\infty(X,Y), \tag{3.8}$$

where $r^\varepsilon_1(X,Y)$, $r^\varepsilon_2(X,Y)$, *and* $\tilde{\kappa}(X,Y)$ *are bounded functions with* $r^\varepsilon_1 \sim O((k\varepsilon)^2\ln(k\varepsilon))$, $r^\varepsilon_2 \sim O((k\varepsilon)^2)$, *and* $\tilde{\kappa}_\infty \sim O(\exp(-1/\varepsilon))$ *for all* $X, Y \in (0,1)$.

Proof. First, from the asymptotic expansion of $H^{(1)}_0$ (cf. [11]), we have

$$\begin{aligned}G^e_\varepsilon(X,Y) &= \left(-\frac{i}{2}\right)H^{(1)}_0(\varepsilon k|X-Y|) \\ &= -\frac{i}{2}\left(\frac{2i}{\pi}\ln(\varepsilon|X-Y|) + \frac{2i}{\pi}\ln k + \frac{2i}{\pi}\gamma_0 + O((k\varepsilon|X-Y|)^2\ln(k\varepsilon|X-Y|))\right) \\ &= \frac{1}{\pi}\left[\ln\varepsilon + \ln|X-Y| + \ln k + \gamma_0 + O\left((k\varepsilon|X-Y|)^2\ln(k\varepsilon|X-Y|)\right)\right].\end{aligned}$$

Recall that

$$G^i_\varepsilon(X,Y) = \frac{1}{\varepsilon}\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} c_{mn}\alpha_{mn}\right)\cos(m\pi X)\cos(m\pi Y). \tag{3.9}$$

Let $C_m = \sum_{n=0}^{\infty} c_{mn}\alpha_{mn}$. Then from the representation of elementary functions by series (cf. [14]), we see that

$$C_0(k) = \sum_{n=1}^{\infty} \frac{2}{k^2-(n\pi)^2} + \frac{1}{k^2} = \frac{\cot k}{k},$$

$$\begin{aligned} C_m(k,\varepsilon) &= \sum_{n=1}^{\infty} \frac{4}{k^2-(m\pi/\varepsilon)^2-(n\pi)^2} + \frac{2}{k^2-(m\pi/\varepsilon)^2} \\ &= -\frac{2}{\sqrt{(m\pi/\varepsilon)^2-k^2}} \coth\left(\sqrt{(m\pi/\varepsilon)^2-k^2}\right) \\ &= -\frac{2\varepsilon}{m\pi} - \frac{k^2\varepsilon^3}{m^3\pi^3} + O\left(\frac{\varepsilon^5}{m^5}\right), \quad m \geq 1. \end{aligned}$$

Substituting into (3.9), we obtain

$$\begin{aligned} G^i_\varepsilon(X,Y) = {}& \frac{1}{\varepsilon}\Bigg\{C_0(k) - \sum_{m\geq 1} \frac{2\varepsilon}{\pi m}\cos(m\pi X)\cos(m\pi Y) - \sum_{m\geq 1}\frac{k^2\varepsilon^3}{m^3\pi^3}\cos(m\pi X)\cos(m\pi Y) \\ & +O\left(\sum_{m\geq 1}\frac{\varepsilon^5}{m^5}\right)\Bigg\} \\ = {}& \frac{\cot k}{k\varepsilon} + \left(-\frac{2}{\pi}\right)\left[-\ln 2 - \frac{1}{2}\ln\left(\left|\sin\left(\frac{\pi(X+Y)}{2}\right)\right|\right) - \frac{1}{2}\ln\left(\left|\sin\left(\frac{\pi(X-Y)}{2}\right)\right|\right)\right] \\ & +O(k^2\varepsilon^2), \end{aligned}$$

and (3.7) follows. Here we have used the following formulas (cf. [15, 17]):

$$\begin{aligned} \sum_{m\geq 1}\frac{1}{m}\cos(m\pi t) &= -\frac{1}{2}\ln\left(4\sin^2\frac{\pi t}{2}\right) \quad 0 < t < 2, \\ \sum_{m\geq 1}\frac{1}{m^3}\cos(m\pi t) &= \sum_{m\geq 1}\frac{1}{m^3} + \frac{(\pi t)^2}{2}\ln(\pi t) + O(t^2) \quad 0 < t < 2. \end{aligned}$$

Finally, note that

$$\tilde{G}^i_\varepsilon(X,Y) = \frac{1}{\varepsilon}\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty}(-1)^n c_{mn}\alpha_{mn}\right)\cos(m\pi X)\cos(m\pi Y).$$

Let $\tilde{C}_m = \sum_{n=0}^{\infty}(-1)^n c_{mn}\alpha_{mn}$. Again the representation of elementary functions as series yields (cf. [14])

$$\tilde{C}_0(k) = \sum_{n=1}^{\infty}\frac{2(-1)^n}{k^2-(n\pi)^2} + \frac{1}{k^2} = \frac{1}{k\sin k},$$

$$
\begin{aligned}
\tilde{C}_m(k,\varepsilon) &= \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 4}{k^2-(m\pi/\varepsilon)^2-(n\pi)^2} + \frac{2}{k^2-(m\pi/\varepsilon)^2} \\
&= -\frac{2}{\sqrt{(m\pi/\varepsilon)^2-k^2}\,\sinh\left(\sqrt{(m\pi/\varepsilon)^2-k^2}\right)} \\
&= O\left(\frac{\varepsilon}{m\pi}\exp(-m\pi/\varepsilon)\right)
\end{aligned}
$$

for $k\varepsilon \ll 1$ and $m \geq 1$. Therefore,

$$
\tilde{G}^i_\varepsilon(X,Y) = \frac{1}{(k\sin k)\varepsilon} + O\left(\exp\left(-1/\varepsilon\right)\right).
$$

The lemma follows. □

Let $\kappa(X,Y)$ and $\tilde{\kappa}_\infty(X,Y)$ be defined in (3.5) and (3.8) respectively. Set $\kappa_\infty(X,Y) = r_1(X,Y)+r_2(X,Y)$, where $r_1(X,Y)$ and $r_2(X,Y)$ are defined in Lemma 3.1. We denote by K, K_∞, $\tilde{K}_\infty$ the integral operators corresponding to the Schwarz kernels $\kappa(X,Y)$, $\kappa_\infty(X,Y)$ and $\tilde{\kappa}_\infty(X,Y)$. We also define the operator $P: V_1 \to V_2$ by

$$
P\varphi(X) = (\varphi, 1)1,
$$

where 1 is a function defined on the interval $(0,1)$ and is equal to one therein. We will use this notation in the sequel. One can easily check that $1 \in V_2$. Thus the above definition is valid.

Lemma 3.2.

(1) The operator $T^e + T^i$ admits the following decomposition:

$$
T^e + T^i = \beta P + K + K_\infty.
$$

Moreover, K_∞ is bounded from V_1 to V_2 with the operator norm $\|K_\infty\| \lesssim \varepsilon^2|\ln\varepsilon|$ uniformly for bounded k.

(2) The operator $\tilde{T}^i$ admits the following decomposition:

$$
\tilde{T}^i = \tilde{\beta} P + \tilde{K}_\infty.
$$

Moreover, $\tilde{K}_\infty$ is bounded from V_1 to V_2 with the operator norm $\|K_\infty\| \lesssim \exp(-1/\varepsilon)$ uniformly for bounded k.

(3) The operator K is bounded from V_1 to V_2 with a bounded inverse. Moreover,

$$
\alpha := \langle K^{-1}1, 1\rangle \neq 0.
$$

The proof of (1) and (2) follows directly from the definition of the operators T^e, T^i and $\tilde{T}^i$ in (2.4) - (2.6) and the asymptotic expansions of their kernels given by Lemma 3.1. The proof of (3) can be found in Theorem 4.1 and Lemma 4.2 of [7].

4. Asymptotic expansions of the resonances. We have shown that the scattering problem (1.1) is equivalent to the system (2.7). A scattering resonance of (1.1) is defined as a complex number k with negative imaginary part such that there is a nontrivial solution to (1.1) when the incident field is zero. This is exactly the characteristic values of the operator-valued function $\mathbb{P}+\mathbb{L}$ with respect to the variable k (see [3] for a systematic treatment of the equivalence), which we seek in what follows.

By virtue of Lemma 3.2, we first write

$$
\begin{bmatrix} T^e + T^i & \tilde{T}^i \\ \tilde{T}^i & T^e + T^i \end{bmatrix} = \begin{bmatrix} \beta P & \tilde{\beta} P \\ \tilde{\beta} P & \beta P \end{bmatrix} + K\mathbb{I} + \begin{bmatrix} K_\infty & \tilde{K}_\infty \\ \tilde{K}_\infty & K_\infty \end{bmatrix} =: \mathbb{P} + \mathbb{L},
$$

where

$$\mathbb{P} = \begin{bmatrix} \beta P & \tilde{\beta} P \\ \tilde{\beta} P & \beta P \end{bmatrix}, \quad \mathbb{K}_\infty = \begin{bmatrix} K_\infty & \tilde{K}_\infty \\ \tilde{K}_\infty & K_\infty \end{bmatrix} \quad \text{and} \quad \mathbb{L} = K\mathbb{I} + \mathbb{K}_\infty.$$

By Lemma 3.2, it is easy to see that $\mathbb{L}$ is invertible for sufficiently small ε. Now assume that there exists $\boldsymbol{\varphi}$ such that

$$(\mathbb{P} + \mathbb{L})\boldsymbol{\varphi} = 0.$$

Then

$$\mathbb{L}^{-1}\, \mathbb{P}\, \boldsymbol{\varphi} + \boldsymbol{\varphi} = 0.$$

Let $\mathbf{e}_1 = [1, 0]^T$ and $\mathbf{e}_2 = [0, 1]^T$. Note that

$$\mathbb{P}\, \boldsymbol{\varphi} = \beta\langle\boldsymbol{\varphi}, \mathbf{e}_1\rangle\mathbf{e}_1 + \beta\langle\boldsymbol{\varphi}, \mathbf{e}_2\rangle\mathbf{e}_2 + \tilde{\beta}\langle\boldsymbol{\varphi}, \mathbf{e}_2\rangle\mathbf{e}_1 + \tilde{\beta}\langle\boldsymbol{\varphi}, \mathbf{e}_1\rangle\mathbf{e}_2.$$

We obtain

$$\beta\langle\boldsymbol{\varphi}, \mathbf{e}_1\rangle\mathbb{L}^{-1}\mathbf{e}_1 + \beta\langle\boldsymbol{\varphi}, \mathbf{e}_2\rangle\mathbb{L}^{-1}\mathbf{e}_2 + \tilde{\beta}\langle\boldsymbol{\varphi}, \mathbf{e}_2\rangle\mathbb{L}^{-1}\mathbf{e}_1 + \tilde{\beta}\langle\boldsymbol{\varphi}, \mathbf{e}_1\rangle\mathbb{L}^{-1}\mathbf{e}_2 + \boldsymbol{\varphi} = 0. \tag{4.1}$$

By taking the inner product of (4.1) with $\mathbf{e}_1$ and $\mathbf{e}_2$ respectively, we have

$$\begin{aligned}
&\beta\langle\boldsymbol{\varphi}, \mathbf{e}_1\rangle\langle\mathbb{L}^{-1}\mathbf{e}_1, \mathbf{e}_1\rangle + \beta\langle\boldsymbol{\varphi}, \mathbf{e}_2\rangle\langle\mathbb{L}^{-1}\mathbf{e}_2, \mathbf{e}_1\rangle + \\
&\tilde{\beta}\langle\boldsymbol{\varphi}, \mathbf{e}_2\rangle\langle\mathbb{L}^{-1}\mathbf{e}_1, \mathbf{e}_1\rangle + \tilde{\beta}\langle\boldsymbol{\varphi}, \mathbf{e}_1\rangle\langle\mathbb{L}^{-1}\mathbf{e}_2, \mathbf{e}_1\rangle + \langle\boldsymbol{\varphi}, \mathbf{e}_1\rangle = 0; \qquad (4.2)\\
&\beta\langle\boldsymbol{\varphi}, \mathbf{e}_1\rangle\langle\mathbb{L}^{-1}\mathbf{e}_1, \mathbf{e}_2\rangle + \beta\langle\boldsymbol{\varphi}, \mathbf{e}_2\rangle\langle\mathbb{L}^{-1}\mathbf{e}_2, \mathbf{e}_2\rangle + \\
&\tilde{\beta}\langle\boldsymbol{\varphi}, \mathbf{e}_2\rangle\langle\mathbb{L}^{-1}\mathbf{e}_1, \mathbf{e}_2\rangle + \tilde{\beta}\langle\boldsymbol{\varphi}, \mathbf{e}_1\rangle\langle\mathbb{L}^{-1}\mathbf{e}_2, \mathbf{e}_2\rangle + \langle\boldsymbol{\varphi}, \mathbf{e}_2\rangle = 0. \qquad (4.3)
\end{aligned}$$

Now, we introduce a matrix $\mathbb{M}$ by letting

$$\begin{aligned}
\mathbb{M} &= \begin{bmatrix} \beta\langle\mathbb{L}^{-1}\mathbf{e}_1, \mathbf{e}_1\rangle + \tilde{\beta}\langle\mathbb{L}^{-1}\mathbf{e}_2, \mathbf{e}_1\rangle & \beta\langle\mathbb{L}^{-1}\mathbf{e}_2, \mathbf{e}_1\rangle + \tilde{\beta}\langle\mathbb{L}^{-1}\mathbf{e}_1, \mathbf{e}_1\rangle\rangle \\ \beta\langle\mathbb{L}^{-1}\mathbf{e}_1, \mathbf{e}_2\rangle + \tilde{\beta}\langle\mathbb{L}^{-1}\mathbf{e}_2, \mathbf{e}_2\rangle & \beta\langle\mathbb{L}^{-1}\mathbf{e}_2, \mathbf{e}_2\rangle + \tilde{\beta}\langle\mathbb{L}^{-1}\mathbf{e}_1, \mathbf{e}_2\rangle \end{bmatrix} \\
&= \beta \begin{bmatrix} \langle\mathbb{L}^{-1}\mathbf{e}_1, \mathbf{e}_1\rangle & \langle\mathbb{L}^{-1}\mathbf{e}_2, \mathbf{e}_1\rangle \\ \langle\mathbb{L}^{-1}\mathbf{e}_1, \mathbf{e}_2\rangle & \langle\mathbb{L}^{-1}\mathbf{e}_2, \mathbf{e}_2\rangle \end{bmatrix} + \tilde{\beta} \begin{bmatrix} \langle\mathbb{L}^{-1}\mathbf{e}_2, \mathbf{e}_1\rangle & \langle\mathbb{L}^{-1}\mathbf{e}_1, \mathbf{e}_1\rangle \\ \langle\mathbb{L}^{-1}\mathbf{e}_2, \mathbf{e}_2\rangle & \langle\mathbb{L}^{-1}\mathbf{e}_1, \mathbf{e}_2\rangle \end{bmatrix}
\end{aligned}$$

Then (4.2) - (4.3) reduce to

$$(\mathbb{M} + \mathbb{I}) \begin{bmatrix} \langle\boldsymbol{\varphi}, \mathbf{e}_1\rangle \\ \langle\boldsymbol{\varphi}, \mathbf{e}_2\rangle \end{bmatrix} = 0.$$

By Lemma 4.2 given later in this section, $\mathbb{M}$ may be rewritten as

$$\mathbb{M} = \beta \begin{bmatrix} \langle\mathbb{L}^{-1}\mathbf{e}_1, \mathbf{e}_1\rangle & \langle\mathbb{L}^{-1}\mathbf{e}_1, \mathbf{e}_2\rangle \\ \langle\mathbb{L}^{-1}\mathbf{e}_1, \mathbf{e}_2\rangle & \langle\mathbb{L}^{-1}\mathbf{e}_1, \mathbf{e}_1\rangle \end{bmatrix} + \tilde{\beta} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \langle\mathbb{L}^{-1}\mathbf{e}_1, \mathbf{e}_1\rangle & \langle\mathbb{L}^{-1}\mathbf{e}_1, \mathbf{e}_2\rangle \\ \langle\mathbb{L}^{-1}\mathbf{e}_1, \mathbf{e}_2\rangle & \langle\mathbb{L}^{-1}\mathbf{e}_1, \mathbf{e}_1\rangle \end{bmatrix}$$

It is straightforward to check that the eigenvalues of $\mathbb{M} + \mathbb{I}$ are

$$\lambda_1(k, \varepsilon) = 1 + (\beta(k, \varepsilon) + \tilde{\beta}(k, \varepsilon))\left(\langle\mathbb{L}^{-1}\mathbf{e}_1, \mathbf{e}_1\rangle + \langle\mathbb{L}^{-1}\mathbf{e}_1, \mathbf{e}_2\rangle\right), \tag{4.4}$$

$$\lambda_2(k, \varepsilon) = 1 + (\beta(k, \varepsilon) - \tilde{\beta}(k, \varepsilon))\left(\langle\mathbb{L}^{-1}\mathbf{e}_1, \mathbf{e}_1\rangle - \langle\mathbb{L}^{-1}\mathbf{e}_1, \mathbf{e}_2\rangle\right). \tag{4.5}$$

The associated eigenvectors are $[1 \quad 1]^T$ and $[1 \quad -1]^T$. Therefore, the characteristic values of the operator-valued function $\mathbb{P}+\mathbb{L}$ are the roots of the two analytic functions

$\lambda_1(k,\varepsilon)$ and $\lambda_2(k,\varepsilon)$, and the associated characteristic functions are given respectively by

$$\boldsymbol{\varphi}_1 = (\beta(k,\varepsilon)+\tilde{\beta}(k,\varepsilon))\cdot(\mathbb{L}^{-1}(\mathbf{e}_1+\mathbf{e}_2)),$$
$$\boldsymbol{\varphi}_2 = (\beta(k,\varepsilon)-\tilde{\beta}(k,\varepsilon))\cdot(\mathbb{L}^{-1}(\mathbf{e}_1-\mathbf{e}_2)).$$

On the other hand, given roots of $\lambda_1(k,\varepsilon)$ and $\lambda_2(k,\varepsilon)$, one can check that they are characteristic values of the operator-valued function $\mathbb{P}+\mathbb{L}$ with corresponding characteristic functions defined above.

In conclusion, we obtain the following lemma.

LEMMA 4.1. *The resonances of the scattering problem* (1.1) *are the roots of the analytic functions* $\lambda_1(k,\varepsilon)=0$ *and* $\lambda_2(k,\varepsilon)=0$.

We now prove a techniqical result which is needed in the derivation of Lemma 4.1.

LEMMA 4.2. *The following identities hold:*

$$\langle \mathbb{L}^{-1}\mathbf{e}_1,\mathbf{e}_1\rangle = \langle \mathbb{L}^{-1}\mathbf{e}_2,\mathbf{e}_2\rangle, \quad \langle \mathbb{L}^{-1}\mathbf{e}_1,\mathbf{e}_2\rangle = \langle \mathbb{L}^{-1}\mathbf{e}_2,\mathbf{e}_1\rangle. \tag{4.6}$$

Moreover,

$$\mathbb{L}^{-1}\mathbf{e}_1 = K^{-1}1\cdot\mathbf{e}_1 + O\left((k\varepsilon)^2\ln(k\varepsilon)\right), \quad \mathbb{L}^{-1}\mathbf{e}_2 = K^{-1}1\cdot\mathbf{e}_2 + O\left((k\varepsilon)^2\ln(k\varepsilon)\right), \tag{4.7}$$

and

$$\langle \mathbb{L}^{-1}\mathbf{e}_1,\mathbf{e}_1\rangle = \alpha + O\left((k\varepsilon)^2\ln(k\varepsilon)\right), \quad \langle \mathbb{L}^{-1}\mathbf{e}_1,\mathbf{e}_2\rangle = O\left((k\varepsilon)^2\ln(k\varepsilon)\right). \tag{4.8}$$

Proof. Let $\mathbb{L}^{-1}\mathbf{e}_1=(a,b)^T$. Then $\mathbb{L}(a,b)^T=\mathbf{e}_1$. More precisely,

$$Ka + K_\infty a + \tilde{K}_\infty b = 1,$$
$$Kb + \tilde{K}_\infty a + K_\infty b = 0.$$

It follows that $\mathbb{L}(b,a)^T=\mathbf{e}_2$, or equivalently,

$$\mathbb{L}^{-1}\mathbf{e}_2 = (b,a)^T,$$

hence the two identities (4.6) follow. Now, applying the Neumann series and the expansions in Lemma 3.1, we obtain

$$\mathbb{L}^{-1} = (K\mathbb{I}+\mathbb{K}_\infty)^{-1} = \left(\sum_{j=0}^{\infty}(-1)^j\left(K^{-1}\mathbb{K}_\infty\right)^j\right)K^{-1} = K^{-1}\mathbb{I} + O\left((k\varepsilon)^2\ln(k\varepsilon)\right).$$

Therefore, (4.7) and (4.8) follow immediately. ◻

Remark The explicit dependence of k for the asymptotic expansions in Lemma 4.2 and several other occasions in the rest of the paper is necessary for the investigation of the quasi-static case. However, such dependence will be omitted for simplicity when k is bounded, especially for the case of resonant scattering.

Finally, we are ready to present the main result of this section, the asymptotic expansion of resonances for the scattering problem (1.1). For clarity, we restrict the discussion on the right half of the complex k-plane, and the resonances on the left half of the complex plane can be derived analogously.

THEOREM 4.3. *Recall that $\alpha = \langle K^{-1}1, 1\rangle$ and $\gamma_0 = c_0 - \ln 2 - i\pi/2$. There exist two sets of resonances, $\{k_{n,1}\}$ and $\{k_{n,2}\}$, for the scattering problem* (1.1)*, and the following asymptotic expansions hold:*

$$k_{n,1} = (2n-1)\pi + (4n-2)\pi\left[\frac{1}{\pi}\varepsilon\ln\varepsilon + \left(\frac{1}{\alpha} + \frac{1}{\pi}\left(2\ln 2 + \ln((2n-1)\pi) + \gamma_0\right)\right)\varepsilon\right] + O(\varepsilon^2\ln^2\varepsilon);$$
$$k_{n,2} = 2n\pi + 4n\pi\left[\frac{1}{\pi}\varepsilon\ln\varepsilon + \left(\frac{1}{\alpha} + \frac{1}{\pi}\left(2\ln 2 + \ln(2n\pi) + \gamma_0\right)\right)\varepsilon\right] + O(\varepsilon^2\ln^2\varepsilon).$$

where $n = 1, 2, 3, \ldots$, and $n\varepsilon \ll 1$.

Proof. . By Lemma 4.1, we find the root of

$$\lambda_1(k,\varepsilon) = 1 + (\beta(k,\varepsilon) + \tilde{\beta}(k,\varepsilon))\left(\langle \mathbb{L}^{-1}\mathbf{e}_1, \mathbf{e}_1\rangle + \langle \mathbb{L}^{-1}\mathbf{e}_1, \mathbf{e}_2\rangle\right) = 0$$

to obtain the first set of resonances. Recall that $\beta(k,\varepsilon) = \dfrac{\cot k}{k\varepsilon} + \dfrac{1}{\pi}(2\ln 2 + \ln k + \gamma_0) + \dfrac{1}{\pi}\ln\varepsilon$ and $\tilde{\beta}(k,\varepsilon) = \dfrac{1}{(k\sin k)\varepsilon}$. We may write the above equation as

$$1+\left[\left(\frac{\cot k}{k} + \frac{1}{k\sin k}\right)\frac{1}{\varepsilon} + \frac{1}{\pi}(2\ln 2 + \ln k + \gamma_0) + \frac{1}{\pi}\ln\varepsilon\right]\left(\langle \mathbb{L}^{-1}\mathbf{e}_1, \mathbf{e}_1\rangle + \langle \mathbb{L}^{-1}\mathbf{e}_1, \mathbf{e}_2\rangle\right) = 0.$$

For simplicity, let us define

$$c(k) = \frac{\cot k}{k} + \frac{1}{k\sin k},$$
$$\gamma(k) = \frac{1}{\pi}(2\ln 2 + \ln k + \gamma_0).$$

We then obtain the following by applying Lemma 4.2,

$$p(k,\varepsilon) := \varepsilon\lambda_1(k,\varepsilon) = \varepsilon + \left[c(k) + \varepsilon\gamma(k) + \frac{1}{\pi}\varepsilon\ln\varepsilon\right](\alpha + r(k,\varepsilon)) = 0, \tag{4.9}$$

where $r(k,\varepsilon)$ is analytic in k and $r(k,\varepsilon) \sim O(\varepsilon^2\ln\varepsilon)$. Note that the negative real axis is the branch cut for $p(k)$, hence we choose a small number $\theta_0 > 0$ and consider the domain $\{z \mid \pi - \theta_0 \le \arg z \le \pi + \theta_0\}$. Here we are only interested in those resonances which are not in the high frequency regime. Therefore, we find all the roots of p in the bounded domain

$$D_{\theta_0,M} = \{z : |z| \le M, -(\pi - \theta_0) \le \arg z \le \pi - \theta_0\}$$

for some fixed number $M > 0$. Observe that p blows up as $k \to 2j\pi$ for all $j \in \mathbb{Z}$, where $\mathbb{Z}$ denotes the set of all non-negative integers. As a result, there exists $\delta_0 > 0$ such that all the root of p in $D_{\theta_0,M}$ actually lies in the smaller domain

$$D_{\delta_0,\theta_0,M} = \{z \mid |z - 2j\pi| \ge \delta_0, \forall j \in \mathbb{Z}\} \cap D_{\theta_0,M}.$$

We see that $p(k,\varepsilon)$ is analytic for k in the domain $D_{\delta_0,\theta_0,M}$.

It is clear that $c(k)$ is analytic in $D_{\delta_0,\theta_0,M}$ and its roots are given by $k_{n,1,0} = (2n-1)\pi$, $n = 1, 2, \cdots$. Note that each root is simple. From Rouche's theorem, we deduce that the roots of $\lambda_1(k,\varepsilon)$, which are denoted as $k_{n,1}$, are also simple and are close to $k_{n,1,0}$'s if ε is sufficiently small. We now derive the leading-order asymptotic terms for these roots.

Define

$$p_1(k,\varepsilon) = \varepsilon + \left[c(k) + \varepsilon\gamma(k) + \frac{1}{\pi}\varepsilon\ln\varepsilon\right]\alpha. \tag{4.10}$$

Then

$$\begin{aligned} p_1(k,\varepsilon) = \varepsilon + \Big[&c'(k_{n,1,0})(k-k_{n,1,0}) + O(k-k_{n,1,0})^2 + \varepsilon\gamma(k_{n,1,0}) \\ &+\varepsilon\cdot O(k-k_{n,1,0}) + \frac{1}{\pi}\varepsilon\ln\varepsilon\Big]\alpha. \end{aligned} \tag{4.11}$$

By a direct calculation, it is seen that $c'(k_{n,1,0}) = -\dfrac{1}{2k_{n,1,0}} = -\dfrac{1}{2(2n-1)\pi}$. We can conclude that p_1 has simple roots in $D_{\delta_0,\theta_0,M}$ which are close to $k_{n,1,0}$'s. Moreover, these roots are analytic with respect to the variable ε and $\varepsilon\ln\varepsilon$. Expanding these roots, which are denoted by $k_{n,1,1}$, in terms of powers of ε and $\varepsilon\ln\varepsilon$, we obtain

$$k_{n,1,1} = k_{n,1,0} + 2k_{n,1,0}\left[\frac{1}{\pi}\varepsilon\ln\varepsilon + \left(\frac{1}{\alpha} + \frac{1}{\pi}(2\ln 2 + \ln((2n-1)\pi) + \gamma_0)\right)\varepsilon\right] + O(\varepsilon^2\ln^2\varepsilon).$$

We claim that $k_{n,1,1}$ gives the leading order for the asymptotic expansion of the roots $k_{n,1}$. More precisely, the following holds:

$$k_{n,1} - k_{n,1,1} = O(\varepsilon^2\ln^2\varepsilon).$$

We prove the claim by using Rouche's theorem. Note that

$$p(k,\varepsilon) - p_1(k,\varepsilon) = O(c(k) + \varepsilon\ln\varepsilon)\, r(k,\varepsilon)$$

and

$$p_1(k,\varepsilon) = c(k)\alpha + O(\varepsilon\ln\varepsilon).$$

One can find a constant $C_n > 0$ such that

$$|p(k,\varepsilon) - p_1(k,\varepsilon)| < |p_1(k,\varepsilon)|$$

for all k such that $|k - k_{n,1,1}| = C_n\varepsilon^2\ln^2\varepsilon$. As a result, p has a simple root in the disc $\{k \mid |k - k_{n,1,1}| = C_n\varepsilon^2\ln^2\varepsilon\}$, which proves our claim.

Similarly, by finding the roots of

$$\lambda_2(k,\varepsilon) = 1 + (\beta(k,\varepsilon) - \tilde\beta(k,\varepsilon))\left(\langle\mathbb{L}^{-1}\mathbf{e}_1,\mathbf{e}_1\rangle - \langle\mathbb{L}^{-1}\mathbf{e}_1,\mathbf{e}_2\rangle\right) = 0$$

in the domain

$$\tilde D_{\delta_0,\theta_0,M} = \{z \mid |z-(2j+1)\pi| \ge \delta_0, \forall j\in\mathbb{Z}\} \cap D_{\theta_0,M},$$

we obtain the second set of resonances. The arguments are the same as above and we omit here. The proof of the theorem is complete. □

In what follows, we would like to refer to $\{k_{n,1}\}$ and $\{k_{n,2}\}$ as odd and even resonances of the scattering problem (1.1) respectively. Now if the thickness of the slab $\ell \neq 1$, applying the scaling argument and the following proposition holds.

Proposition 4.4. *Let ℓ be the thickness of the slab, then the following asymptotic expansions hold for the resonances of the scattering problem* (1.1)*:*

$$k_{n,1}\ell = (2n-1)\pi + (4n-2)\pi \left[\frac{1}{\pi}\varepsilon \ln\varepsilon + \left(\frac{1}{\alpha} + \frac{1}{\pi}(2\ln 2 + \ln((2n-1)\pi) + \gamma_0)\right)\varepsilon\right] + O(\varepsilon^2 \ln^2 \varepsilon);$$
$$k_{n,2}\ell = 2n\pi + 4n\pi \left[\frac{1}{\pi}\varepsilon \ln\varepsilon + \left(\frac{1}{\alpha} + \frac{1}{\pi}(2\ln 2 + \ln(2n\pi) + \gamma_0)\right)\varepsilon\right] + O(\varepsilon^2 \ln^2 \varepsilon).$$

where $n = 1, 2, 3, ...,$ *and* $n\varepsilon \ll 1$.

5. Quantitative analysis of field enhancement at resonant frequencies.

5.1. Preliminaries. To investigate the field enhancement, in this section we present some preliminary calculations and study the solution of the system (2.7). First recall that $p(k,\varepsilon) = \varepsilon\lambda_1(k,\varepsilon)$ and is given by

$$p(k,\varepsilon) = \varepsilon + \left[\frac{\cot k}{k} + \frac{1}{k\sin k} + \frac{\varepsilon}{\pi}(2\ln 2 + \ln k + \gamma_0) + \frac{1}{\pi}\varepsilon\ln\varepsilon\right](\alpha + r(k,\varepsilon)). \tag{5.1}$$

Similarly we define $q(k,\varepsilon) := \varepsilon\lambda_2(k,\varepsilon)$, or more explicitly,

$$q(k,\varepsilon) = \varepsilon + \left[\frac{\cot k}{k} - \frac{1}{k\sin k} + \frac{\varepsilon}{\pi}(2\ln 2 + \ln k + \gamma_0) + \frac{1}{\pi}\varepsilon\ln\varepsilon\right](\alpha + s(k,\varepsilon)), \tag{5.2}$$

where $s(k,\varepsilon) \sim O(\varepsilon^2 \ln\varepsilon)$.

Lemma 5.1. *If* $n\varepsilon \ll 1$*, then at the odd and even resonant frequencies* $k = \mathrm{Re}\, k_{n,1}$ *and* $k = \mathrm{Re}\, k_{n,2}$*, we have*

$$p(k,\varepsilon) = -\frac{i\alpha}{2}\varepsilon + O(\varepsilon^2\ln^2\varepsilon) \quad \textit{and} \quad q(k,\varepsilon) = -\frac{i\alpha}{2}\varepsilon + O(\varepsilon^2\ln^2\varepsilon)$$

respectively.

Proof. We first consider $p(k,\varepsilon)$. Assume that

$$|k - \mathrm{Re}\, k_{n,1}| \le \varepsilon|\ln\varepsilon|.$$

From the definition of p_1 in (4.10) and its expansion (4.11), it follows that

$$\begin{aligned} p(k,\varepsilon) &= p_1(k,\varepsilon) + O(\varepsilon^2\ln\varepsilon) \\ &= p_1'(k_{n,1})(k - k_{n,1}) + O(k-k_{n,1})^2 + O(\varepsilon^2\ln\varepsilon) \\ &= [\alpha c'(k_{n,1}) + O(\varepsilon\ln\varepsilon)]\cdot(k-k_{n,1}) + O(\varepsilon^2\ln^2\varepsilon) \\ &= \alpha c'(k_{n,1,0})\cdot(k-k_{n,1}) + O(\varepsilon^2\ln^2\varepsilon) \\ &= -\frac{\alpha}{2(2n-1)\pi}(k - \mathrm{Re}\, k_{n,1} - i\,\mathrm{Im}\, k_{n,1}) + O(\varepsilon^2\ln^2\varepsilon). \end{aligned}$$

Note that

$$\mathrm{Im}\, k_{n,1} = \mathrm{Im}\, k_{n,1,1} + O(\varepsilon^2\ln^2\varepsilon) = -(2n-1)\pi\varepsilon + O(\varepsilon^2\ln^2\varepsilon).$$

We deduce that at the odd resonant frequencies $k = \mathrm{Re}\, k_{n,1}$,

$$p(k,\varepsilon) = -\frac{i\alpha}{2}\varepsilon + O(\varepsilon^2\ln^2\varepsilon).$$

The calculations for $q(k,\varepsilon)$ at the even resonant frequencies $k = \operatorname{Re} k_{n,2}$ are the same. $\square$

LEMMA 5.2. *Recall that the incidence direction $d = (d_1, -d_2)$. The following asymptotic expansion holds for the solution $\boldsymbol{\varphi}$ of (2.7) in $V_1 \times V_1$:*

$$\begin{aligned}\boldsymbol{\varphi} = K^{-1}1 \cdot \left[d_1 \cdot O(k) \cdot \mathbf{e}_1 + \frac{\alpha}{p}(\mathbf{e}_1 + \mathbf{e}_2) + \frac{\alpha}{q}(\mathbf{e}_1 - \mathbf{e}_2) \right] \\ + \left(\frac{\alpha}{p} + \frac{\alpha}{q} \right) \left[d_1 \cdot O(k\varepsilon) + O(k^2\varepsilon^2 \ln(k\varepsilon)) \right] + O(k^2 \varepsilon \ln(k\varepsilon)).\end{aligned} \tag{5.3}$$

Moreover,

$$\begin{bmatrix} \langle \boldsymbol{\varphi}, \mathbf{e}_1 \rangle \\ \langle \boldsymbol{\varphi}, \mathbf{e}_2 \rangle \end{bmatrix} = \left[\alpha + d_1 \cdot O(k\varepsilon) + O(k^2\varepsilon^2 \ln(k\varepsilon)) \right] \left(\frac{1}{p} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{q} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right). \tag{5.4}$$

Proof. Let $\mathbf{f} = [f/\varepsilon, 0]$, then (2.7) is equivalent to

$$\mathbb{L}^{-1}\, \mathbb{P}\, \boldsymbol{\varphi} + \boldsymbol{\varphi} = \mathbb{L}^{-1}\mathbf{f}.$$

By a calculation similar to that in the previous section, we have

$$\begin{bmatrix} \langle \boldsymbol{\varphi}, \mathbf{e}_1 \rangle \\ \langle \boldsymbol{\varphi}, \mathbf{e}_2 \rangle \end{bmatrix} = (\mathbb{M} + \mathbb{I})^{-1} \begin{bmatrix} \langle \mathbb{L}^{-1}\mathbf{f}, \mathbf{e}_1 \rangle \\ \langle \mathbb{L}^{-1}\mathbf{f}, \mathbf{e}_2 \rangle \end{bmatrix},$$

and

$$\boldsymbol{\varphi} = \mathbb{L}^{-1}\mathbf{f} - \left[\mathbb{L}^{-1}\mathbf{e}_1 \quad \mathbb{L}^{-1}\mathbf{e}_2 \right] \begin{bmatrix} \beta(k,\varepsilon) & \tilde{\beta}(k,\varepsilon) \\ \tilde{\beta}(k,\varepsilon) & \beta(k,\varepsilon) \end{bmatrix} (\mathbb{M} + \mathbb{I})^{-1} \begin{bmatrix} \langle \mathbb{L}^{-1}\mathbf{f}, \mathbf{e}_1 \rangle \\ \langle \mathbb{L}^{-1}\mathbf{f}, \mathbf{e}_2 \rangle \end{bmatrix}.$$

Recall that the matrix $\mathbb{M} + \mathbb{I}$ has two eigenvalues $\lambda_1(k,\varepsilon)$ and $\lambda_2(k,\varepsilon)$, which are associated with the eigenvectors $[1 \quad 1]^T$ and $[1 \quad -1]^T$ respectively. Thus

$$(\mathbb{M} + \mathbb{I})^{-1} = \frac{1}{2\lambda_1(k,\varepsilon)} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{1}{2\lambda_2(k,\varepsilon)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

It follows that

$$\begin{bmatrix} \langle \boldsymbol{\varphi}, \mathbf{e}_1 \rangle \\ \langle \boldsymbol{\varphi}, \mathbf{e}_2 \rangle \end{bmatrix} = \frac{1}{2\lambda_1(k,\varepsilon)} \langle \mathbb{L}^{-1}\mathbf{f}, \mathbf{e}_1 + \mathbf{e}_2 \rangle \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2\lambda_2(k,\varepsilon)} \langle \mathbb{L}^{-1}\mathbf{f}, \mathbf{e}_1 - \mathbf{e}_2 \rangle \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

and

$$\begin{aligned}\boldsymbol{\varphi} = \mathbb{L}^{-1}\mathbf{f} - \frac{1}{2\lambda_1(k,\varepsilon)} \left[\mathbb{L}^{-1}\mathbf{e}_1 \quad \mathbb{L}^{-1}\mathbf{e}_2 \right] \begin{bmatrix} \beta(k,\varepsilon) & \tilde{\beta}(k,\varepsilon) \\ \tilde{\beta}(k,\varepsilon) & \beta(k,\varepsilon) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \langle \mathbb{L}^{-1}\mathbf{f}, \mathbf{e}_1 \rangle \\ \langle \mathbb{L}^{-1}\mathbf{f}, \mathbf{e}_2 \rangle \end{bmatrix} \\ - \frac{1}{2\lambda_2(k,\varepsilon)} \left[\mathbb{L}^{-1}\mathbf{e}_1 \quad \mathbb{L}^{-1}\mathbf{e}_2 \right] \begin{bmatrix} \beta(k,\varepsilon) & \tilde{\beta}(k,\varepsilon) \\ \tilde{\beta}(k,\varepsilon) & \beta(k,\varepsilon) \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \langle \mathbb{L}^{-1}\mathbf{f}, \mathbf{e}_1 \rangle \\ \langle \mathbb{L}^{-1}\mathbf{f}, \mathbf{e}_2 \rangle \end{bmatrix}.\end{aligned}$$

A further calculation yields

$$\begin{aligned}\boldsymbol{\varphi} = \mathbb{L}^{-1}\mathbf{f} + \frac{1 - \lambda_1(k,\varepsilon)/(\mathbb{L}^{-1}\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2)}{2\lambda_1(k,\varepsilon)} \langle \mathbb{L}^{-1}\mathbf{f}, \mathbf{e}_1 + \mathbf{e}_2 \rangle \cdot (\mathbb{L}^{-1}\mathbf{e}_1 + \mathbb{L}^{-1}\mathbf{e}_2) \\ + \frac{1 - \lambda_2(k,\varepsilon)/(\mathbb{L}^{-1}\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2)}{2\lambda_2(k,\varepsilon)} \langle \mathbb{L}^{-1}\mathbf{f}, \mathbf{e}_1 - \mathbf{e}_2 \rangle \cdot (\mathbb{L}^{-1}\mathbf{e}_1 - \mathbb{L}^{-1}\mathbf{e}_2).\end{aligned}$$

It is easy to check that

$$\mathbf{f} = \frac{1}{\varepsilon} 2 \cdot \mathbf{e}_1 + d_1 \cdot O(k) \cdot \mathbf{e}_1, \quad \text{in } V_2 \times V_2.$$

Thus

$$\mathbb{L}^{-1}\mathbf{f} = \frac{1}{\varepsilon}\left[2 + d_1 \cdot O(k\varepsilon)\right]\left[K^{-1}1 \cdot \mathbf{e}_1 + O(k^2\varepsilon^2 \ln(k\varepsilon))\right].$$

Combined with Lemma 4.2, we get

$$\begin{aligned}
\varepsilon\boldsymbol{\varphi} &= \left[2 + d_1 \cdot O(k\varepsilon)\right] K^{-1}1 \cdot \mathbf{e}_1 + O(k^2\varepsilon^2 \ln(k\varepsilon)) \\
&\quad + \frac{1 - \lambda_1/(\alpha + O(k^2\varepsilon^2 \ln(k\varepsilon)))}{2\lambda_1} \cdot \left[2\alpha + d_1 \cdot O(k\varepsilon) + O(k^2\varepsilon^2 \ln(k\varepsilon))\right] \\
&\quad \cdot \left[K^{-1}1 \cdot (\mathbf{e}_1 + \mathbf{e}_2) + O(k^2\varepsilon^2 \ln(k\varepsilon))\right] \\
&\quad + \frac{1 - \lambda_2/(\alpha + O(k^2\varepsilon^2 \ln(k\varepsilon)))}{2\lambda_2} \cdot \left[2\alpha + d_1 \cdot O(k\varepsilon) + O(k^2\varepsilon^2 \ln(k\varepsilon))\right] \\
&\quad \cdot \left[K^{-1}1 \cdot (\mathbf{e}_1 - \mathbf{e}_2) + O(k^2\varepsilon^2 \ln(k\varepsilon))\right] \\
&= d_1 \cdot O(k\varepsilon) \cdot K^{-1}1 \cdot \mathbf{e}_1 + \frac{\alpha}{\lambda_1}\left[K^{-1}1 \cdot (\mathbf{e}_1 + \mathbf{e}_2) + d_1 \cdot O(k\varepsilon) + O(k^2\varepsilon^2 \ln(k\varepsilon))\right] \\
&\quad + \frac{\alpha}{\lambda_2}\left[K^{-1}1 \cdot (\mathbf{e}_1 - \mathbf{e}_2) + d_1 \cdot O(k\varepsilon) + O(k^2\varepsilon^2 \ln(k\varepsilon))\right] + O(k^2\varepsilon^2 \ln(k\varepsilon)),
\end{aligned}$$

hence (5.3) follows. Similarly, we can deduce (5.4). This completes the proof of the lemma. ☐

The following proposition follows directly from Lemma 5.1 and 5.2.

PROPOSITION 5.3. *At resonant frequencies, $\boldsymbol{\varphi} \sim O(1/\varepsilon)$ in $V_1 \times V_1$ and $\langle \boldsymbol{\varphi}, \mathbf{e}_i \rangle \sim O(1/\varepsilon)$, i=1,2.*

5.2. Scattering enhancement in the far field. We first investigate the scattered field in the domain $\Omega^+ \backslash D_1^+$ above the slit, where $D_1^+ := \{x \mid |x - (0,1)| \le 1\}$. Recall that

$$u_\varepsilon^s(x) = \int_{\Gamma_\varepsilon^+} g^e(k; x, y) \frac{\partial u_\varepsilon(y)}{\partial \nu} ds_y, \quad x \in \Omega^+,$$

and

$$\frac{\partial u_\varepsilon}{\partial \nu}(x_1, 1) = -\varphi_1\left(\frac{x_1}{\varepsilon}\right).$$

Thus

$$u_\varepsilon^s(x) = -\int_{\Gamma_\varepsilon^+} g^e(x, (y_1, 1))\varphi_1\left(\frac{y_1}{\varepsilon}\right) dy_1 = -\varepsilon \int_0^1 g^e(x, (\varepsilon Y, 1))\varphi_1(Y) dY.$$

Since

$$g^e(x, (\varepsilon Y, 1)) = g^e(x, (0, 1))\left(1 + O(\varepsilon)\right) \quad \text{for } x \in \Omega^+ \backslash D_1^+,$$

and, by Lemma 5.2,

$$\int_0^1 \varphi_1(Y) dY = \langle \boldsymbol{\varphi}, \mathbf{e}_1 \rangle = \left(\alpha + d_1 \cdot O(\varepsilon) + O(\varepsilon^2 \ln \varepsilon)\right) \cdot \left(\frac{1}{p} + \frac{1}{q}\right).$$

It follows that

$$
\begin{aligned}
u^s_\varepsilon(x) &= -\varepsilon g^e(x,(0,1))(1+O(\varepsilon))\int_0^1 \varphi_1(Y)dY \\
&= -\varepsilon\alpha g^e(x,(0,1))\cdot\left(\frac{1}{p}+\frac{1}{q}\right)+O(\varepsilon^2)\cdot\left(\frac{1}{p}+\frac{1}{q}\right).
\end{aligned}
$$

It is clear that in the far field, the scattered field behaves as the radiating field of a monopole located at $(1,0)$ and with an amplitude of $\varepsilon\,|\alpha|\left|\frac{1}{p}+\frac{1}{q}\right|$. The contribution due to resonance comes from the terms $\frac{1}{p}$ and $\frac{1}{q}$, where the former and the latter corresponds to the enhancement order at the odd and even resonant frequencies respectively. More precisely, an application of Lemma 5.1 yields

$$
\frac{1}{p}=\frac{2i}{\alpha\varepsilon}(1+O(\varepsilon\ln^2\varepsilon)),
$$

at the odd resonant frequencies $k=\operatorname{Re}k_{n,1}$. Correspondingly, the scattered field

$$
u^s_\varepsilon(x)=-2i\cdot g^e(x,(0,1))+O(\varepsilon\ln^2\varepsilon),
$$

which shows that the scattering enhancement is of order $O(1/\varepsilon)$ compared to the $O(\varepsilon)$ order for the scattered field at non-resonance frequencies. The same occurs at the even resonant frequencies $k=\operatorname{Re}k_{n,2}$ by an application of Lemma 5.1. This demonstrates that the scattering enhancement due to Fabry-Perot type resonances.

Following the same lines above, it can be shown that in the far field zone $\Omega^-\backslash D_1^-$ below the slit, where $D_1^-:=\{x\mid |x-(0,0)|\le 1\}$, the transmitted field is equivalent to the radiating field of a monopole located at $(0,0)$. That is,

$$
u^s_\varepsilon(x)=-\varepsilon\alpha g^e(x,(0,0))\cdot\left(\frac{1}{p}-\frac{1}{q}\right)+O(\varepsilon^2)\cdot\left(\frac{1}{p}-\frac{1}{q}\right).
$$

It follows that

$$
u^s_\varepsilon(x)=-2i\cdot g^e(x,(0,1))+O(\varepsilon\ln^2\varepsilon)\quad\text{and}\quad u^s_\varepsilon(x)=2i\cdot g^e(x,(0,1))+O(\varepsilon\ln^2\varepsilon)
$$

at the odd and even resonant frequencies respectively. Again the transmission enhancement is of order $O(1/\varepsilon)$ at the resonant frequencies.

Finally, since the magnetic field $H_\varepsilon=[0,0,u_\varepsilon]$. From the Ampere's law $\nabla\times H_\varepsilon=-i\omega\tau_0E_\varepsilon$, where τ_0 is the electric permittivity in the vacuum, it is straightforward that the electric field enhancement is also of oder $O(1/\varepsilon)$ in the far field zone at the resonant frequencies.

5.3. Field enhancement in the slit. We now investigate the field inside the slit. Note that u_ε satisfies

$$
\begin{cases}
\Delta u_\varepsilon+k^2u_\varepsilon=0 \quad \text{in } S_\varepsilon,\\
\dfrac{\partial u_\varepsilon}{\partial x_1}=0 \quad \text{on } x_1=0,\ x_1=\varepsilon.
\end{cases}
$$

If $k\varepsilon \ll 1$, in light of the boundary condition on the slit walls, we may expand u_ε as the sum of wave-guide modes as follows:

$$(5.5)\quad u_\varepsilon(x) = a_0 \cos kx_2 + b_0 \cos k(1-x_2) + \sum_{m\geq 1} a_m \cos\frac{m\pi x_1}{\varepsilon}\exp\left(-k_2^{(m)}x_2\right) + \sum_{m\geq 1} b_m \cos\frac{m\pi x_1}{\varepsilon}\exp\left(-k_2^{(m)}(1-x_2)\right),$$

where $k_2^{(m)} = \sqrt{(m\pi/\varepsilon)^2 - k^2}$.

Lemma 5.4. *The following holds for the expansion coefficients in* (5.5)*:*

$$a_0 = \frac{1}{k\sin k}\left[\alpha + d_1\cdot O(k\varepsilon) + O(k^2\varepsilon^2\ln(k\varepsilon))\right]\left(\frac{1}{p}+\frac{1}{q}\right),$$

$$b_0 = \frac{1}{k\sin k}\left[\alpha + d_1\cdot O(\varepsilon) + O(k^2\varepsilon^2\ln(k\varepsilon))\right]\left(\frac{1}{p}-\frac{1}{q}\right),$$

$$\sqrt{m}\,|a_m| \leq C, \quad \sqrt{m}\,|b_m| \leq C, \quad \textit{for } m\geq 1,$$

where C is some positive constant independent of ε, k and m.

Proof. Taking the derivative of (5.5), it follows that

$$\frac{\partial u_\varepsilon}{\partial x_2} = -a_0 k\sin kx_2 + b_0 k \sin k(1-x_2) - \sum_{m\geq 1} a_m k_2^{(m)} \cos\frac{m\pi x_1}{\varepsilon}\exp\left(-k_2^{(m)}x_2\right) + \sum_{m\geq 1} b_m k_2^{(m)} \cos\frac{m\pi x_1}{\varepsilon}\exp\left(-k_2^{(m)}(1-x_2)\right).$$

Especially, we have

$$\frac{\partial u_\varepsilon}{\partial x_2}(x_1,1) = -a_0 k\sin k + \sum_{m\geq 1}\left(-a_m\exp\left(-k_2^{(m)}\right)+b_m\right)k_2^{(m)}\cos\frac{m\pi x_1}{\varepsilon},$$
$$\frac{\partial u_\varepsilon}{\partial x_2}(x_1,0) = b_0 k\sin k + \sum_{m\geq 1}\left(-a_m + b_m\exp\left(-k_2^{(m)}\right)\right)k_2^{(m)}\cos\frac{m\pi x_1}{\varepsilon}.$$

It is clear that

$$\begin{aligned}
-a_0 k\sin k &= \frac{1}{\varepsilon}\int_{\Gamma_\varepsilon^+}\frac{\partial u_\varepsilon}{\partial x_2}(x_1,1)dx_1 = \frac{1}{\varepsilon}\int_0^\varepsilon -\varphi_1\left(\frac{x_1}{\varepsilon}\right)dx_1 = -\int_0^1 \varphi_1(X)dX\\
&= -\left[\alpha + d_1\cdot O(k\varepsilon) + O(k^2\varepsilon^2\ln(k\varepsilon))\right]\left(\frac{1}{p}+\frac{1}{q}\right),\\
b_0 k\sin k &= \frac{1}{\varepsilon}\int_{\Gamma_\varepsilon^-}\frac{\partial u_\varepsilon}{\partial x_2}(x_1,0)dx_1 = \frac{1}{\varepsilon}\int_0^\varepsilon \varphi_2\left(\frac{x_1}{\varepsilon}\right)dx_1 = \int_0^1 \varphi_2(X)dX\\
&= \left[\alpha + d_1\cdot O(k\varepsilon) + O(k^2\varepsilon^2\ln(k\varepsilon))\right]\left(\frac{1}{p}-\frac{1}{q}\right),
\end{aligned}$$

and the formulas for a_0 and b_0 follow.

For $m \geq 1$, it is seen that

$$\begin{cases} \left(-a_m \exp\left(-k_2^{(m)}\right) + b_m\right) k_2^{(m)} = \dfrac{2}{\varepsilon} \displaystyle\int_{\Gamma_\varepsilon^+} \dfrac{\partial u_\varepsilon}{\partial x_2}(x_1, 1) \cos \dfrac{m\pi x_1}{\varepsilon} dx_1 = -2 \int_0^1 \varphi_1(X) \cos(m\pi X) dX, \\ \left(-a_m + b_m \exp\left(-k_2^{(m)}\right)\right) k_2^{(m)} = \dfrac{2}{\varepsilon} \displaystyle\int_{\Gamma_\varepsilon^-} \dfrac{\partial u_\varepsilon}{\partial x_2}(x_1, 0) \cos \dfrac{m\pi x_1}{\varepsilon} dx_1 = 2 \int_0^1 \varphi_2(X) \cos(m\pi X) dX. \end{cases}$$

A straightforward calculation yields

$$a_m k_2^{(m)} = \frac{-2}{1 - \exp\left(-2k_2^{(m)}\right)} \left[\exp\left(-k_2^{(m)}\right) \int_0^1 \varphi_1(X) \cos(m\pi X) dX + \int_0^1 \varphi_2(X) \cos(m\pi X) dX\right],$$

$$b_m k_2^{(m)} = \frac{-2}{1 - \exp\left(-2k_2^{(m)}\right)} \left[\int_0^1 \varphi_1(X) \cos(m\pi X) dX + \exp\left(-k_2^{(m)}\right) \int_0^1 \varphi_2(X) \cos(m\pi X) dX\right].$$

Note that $k_2^{(m)} = O(\frac{m}{\varepsilon})$ for $m \geq 1$, and

$$\|\varphi_1\|_{V_1} \lesssim \frac{1}{\varepsilon}, \quad \|\varphi_2\|_{V_1} \lesssim \frac{1}{\varepsilon}, \quad \|\cos(m\pi X)\|_{V_2} \lesssim \sqrt{m}.$$

The desired estimates for a_m and b_m follow immediately. $\square$

The shape of resonant wave modes in the slit and their enhancement orders are characterized in the following theorem.

THEOREM 5.5. *The wave field in the slit region* $S_\varepsilon^{int} := \{x \in S_\varepsilon \mid x_2 \gg \varepsilon, 1 - x_2 \gg \varepsilon\}$ *is given by*

$$u_\varepsilon(x) = \left(\frac{1}{\varepsilon} + O(\ln^2 \varepsilon) + d_1 \cdot O(1)\right) \cdot \frac{2i}{k \sin(k/2)} \cdot \cos(k(x_2 - 1/2)) + \frac{\sin(k(x_2 - 1/2))}{\sin(k/2)} + O(\varepsilon \ln \varepsilon)$$

and

$$u_\varepsilon(x) = -\left(\frac{1}{\varepsilon} + O(\ln^2 \varepsilon) + d_1 \cdot O(1)\right) \cdot \frac{2i}{k \cos(k/2)} \cdot \sin(k(x_2 - 1/2)) + \frac{\cos(k(x_2 - 1/2))}{\cos(k/2)} + O(\varepsilon \ln \varepsilon)$$

at the resonant frequencies $k = Re\, k_{n,1}$ *and* $k = Re\, k_{n,2}$ *respectively.*

Proof. From the expansion (5.5) and Lemma 5.4, we see that in the region S_ε^{int},

$$\begin{aligned} u_\varepsilon(x) &= \left[\alpha + d_1 \cdot O(\varepsilon) + O(\varepsilon^2 \ln \varepsilon)\right] \left[\frac{\cos(kx_2)}{k \sin k} \left(\frac{1}{p} + \frac{1}{q}\right) + \frac{\cos(k(1 - x_2))}{k \sin k} \left(\frac{1}{p} - \frac{1}{q}\right)\right] \\ &\quad + O\left(\exp\left(-1/\varepsilon\right)\right) \\ &= 2\left[\alpha + d_1 \cdot O(\varepsilon) + O(\varepsilon^2 \ln \varepsilon)\right] \left[\frac{1}{p} \frac{\cos(k/2) \cos(k(x_2 - 1/2))}{k \sin k} - \frac{1}{q} \frac{\sin(k/2) \sin(k(x_2 - 1/2))}{k \sin k}\right] \\ &\quad + O\left(\exp\left(-1/\varepsilon\right)\right). \end{aligned}$$

Now at the odd resonant frequencies $k = \mathrm{Re}\, k_{n,1}$, note that

$$\frac{1}{p} = \frac{2i}{\alpha\varepsilon}(1 + O(\varepsilon \ln^2 \varepsilon)) \quad \text{and} \quad \frac{1}{q} = \frac{k \sin k}{(\cos k - 1)\alpha}(1 + O(\varepsilon \ln \varepsilon)),$$

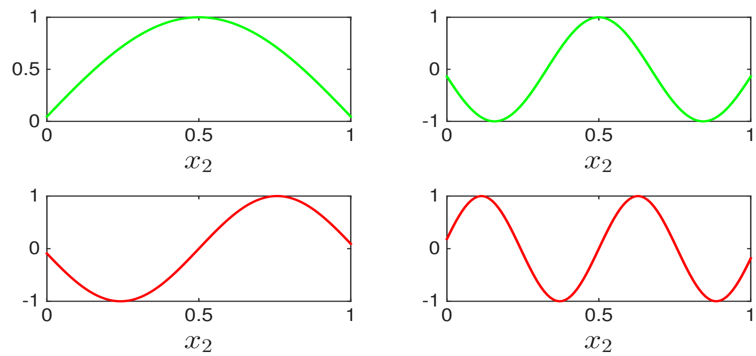

FIG. 5.1. *The shape of the leading term along x_2 for the first two odd resonant modes (Top) and the first two even resonant modes (Bottom) when $\varepsilon = 0.01$. It is calculated that the first two odd resonant frequencies are $k_{1,1} \approx 3.05$ and $k_{2,1} \approx 9.15$, and the first two even resonant frequencies are $k_{1,2} \approx 6.10$ and $k_{2,2} \approx 12.20$. Note that the leading term of resonant modes are constant along the x_1 direction.*

we obtain

$$\begin{aligned}
u_\varepsilon(x) &= \left(1 + d_1 \cdot O(\varepsilon) + O(\varepsilon^2 \ln \varepsilon)\right) \left[\frac{1}{\varepsilon} \cdot \frac{2i\cos(k(x_2 - 1/2))}{k \sin(k/2)}(1 + O(\varepsilon \ln^2 \varepsilon)) \right. \\
&\quad \left. + \frac{\sin(k(x_2 - 1/2))}{\sin(k/2)}(1 + O(\varepsilon \ln \varepsilon))\right] + O\left(\exp\left(-1/\varepsilon\right)\right) \\
&= \left(\frac{1}{\varepsilon} + O(\ln^2 \varepsilon) + d_1 \cdot O(1)\right) \cdot \frac{2i}{k \sin(k/2)} \cdot \cos(k(x_2 - 1/2)) \\
&\quad + \frac{\sin(k(x_2 - 1/2))}{\sin(k/2)} + O(\varepsilon \ln \varepsilon).
\end{aligned}$$

Similarly, at even resonant frequencies,

$$\frac{1}{p} = \frac{k \sin k}{(\cos k + 1)\alpha}(1 + O(\varepsilon \ln \varepsilon)) \quad \text{and} \quad \frac{1}{q} = \frac{2i}{\alpha\varepsilon}(1 + O(\varepsilon \ln^2 \varepsilon)),$$

and we have

$$\begin{aligned}
u_\varepsilon(x) &= \left(1 + d_1 \cdot O(\varepsilon) + O(\varepsilon^2 \ln \varepsilon)\right) \left[\frac{\cos(k(x_2 - 1/2))}{k \cos(k/2)}(1 + O(\varepsilon \ln \varepsilon)) \right. \\
&\quad \left. - \frac{1}{\varepsilon} \cdot \frac{2i\sin(k(x_2 - 1/2))}{k \cos(k/2)}(1 + O(\varepsilon \ln^2 \varepsilon))\right] + O\left(\exp\left(-1/\varepsilon\right)\right) \\
&= -\left(\frac{1}{\varepsilon} + O(\ln^2 \varepsilon) + d_1 \cdot O(1)\right) \cdot \frac{2i}{k \cos(k/2)} \cdot \sin(k(x_2 - 1/2)) \\
&\quad + \frac{\cos(k(x_2 - 1/2))}{\cos(k/2)} + O(\varepsilon \ln \varepsilon).
\end{aligned}$$

$\square$

Therefore, the enhancement due to the resonance is of order $O(1/\varepsilon)$ in the slit. Moreover, the dominant resonant modes in the slit takes the surprisingly simple form of $\cos(k(x_2 - 1/2))$ and $\sin(k(x_2 - 1/2))$ at $k = \operatorname{Re} k_{n,1}$ and $k = \operatorname{Re} k_{n,2}$ respectively. This is illustrated in Figure 5.1. We also remark that the electric field enhancement is also of order $O(1/\varepsilon)$ in the slit, as observed from the Ampere's law.

5.4. Field enhancement on the slit apertures. We now consider the field enhancement on the two apertures Γ_ε^+ and Γ_ε^-. Recall that on Γ_ε^+,

$$u_\varepsilon(x) = \int_{\Gamma_\varepsilon^+} g_\varepsilon^e(x,y)\frac{\partial u_\varepsilon(y)}{\partial \nu} ds_y + u^i + u^r.$$

Let $x_1 = \varepsilon X$, $y_1 = \varepsilon Y$. We have

$$u_\varepsilon(\varepsilon X, 1) = -\int_0^1 G_\varepsilon^e(X,Y)\varepsilon\varphi_1(Y)dY + f(X).$$

Using Lemma 5.2 and the asymptotic expansion of $G_\varepsilon^e(X,Y)$ in Lemma 3.1, we obtain

$$\begin{aligned}
u_\varepsilon(\varepsilon X, 1) &= -\varepsilon\beta_1(k,\varepsilon)\Big(\alpha + d_1\cdot O(k\varepsilon) + O(k^2\varepsilon^2\ln(k\varepsilon))\Big)\left(\frac{1}{p}+\frac{1}{q}\right) \\
&\quad -\frac{\varepsilon}{\pi}\left(d_1\cdot O(k) + \frac{\alpha}{p}+\frac{\alpha}{q}\right)\int_0^1 \ln|X-Y|(K^{-1}1)(Y)dY \\
&\quad -\varepsilon\left(\frac{\alpha}{p}+\frac{\alpha}{q}\right)(d_1\cdot O(k\varepsilon)) + O(k^2\varepsilon^2\ln(k\varepsilon))) + O(k^2\varepsilon^2\ln(k\varepsilon)) + f(X) \\
&= -\varepsilon\cdot\left(\frac{\alpha}{p}+\frac{\alpha}{q}\right)\cdot\left(\beta_1(k,\varepsilon) + \frac{1}{\pi}\int_0^1 \ln|X-Y|(K^{-1}1)(Y)dY\right) + f(X) \\
&\quad -\varepsilon\cdot\left(\beta_1(k,\varepsilon)+\frac{\alpha}{p}+\frac{\alpha}{q}\right)\big(d_1\cdot O(k\varepsilon) + O(k^2\varepsilon^2\ln(k\varepsilon))\big) - d_1\cdot O(k\varepsilon) + O(k^2\varepsilon^2\ln(k\varepsilon)).
\end{aligned}$$

Let us define

$$h(X) = \int_0^1 \ln|X-Y|(K^{-1}1)(Y)dY. \tag{5.6}$$

Using $\beta_1(k,\varepsilon) = \frac{1}{\pi}(\ln k + \gamma_0) + \frac{1}{\pi}\ln\varepsilon$, it is seen that

$$\begin{aligned}
u_\varepsilon(\varepsilon X, 1) &= -\frac{1}{\pi}\left(\frac{\alpha}{p}+\frac{\alpha}{q}\right)\cdot\varepsilon\ln(k\varepsilon) - \frac{1}{\pi}\left(\frac{\alpha}{p}+\frac{\alpha}{q}\right)(\gamma_0 + h(X))\cdot\varepsilon + f(X) \\
&\quad -\left(\frac{\ln(k\varepsilon)}{\pi}+\frac{\alpha}{p}+\frac{\alpha}{q}\right)\cdot\varepsilon\cdot\big(d_1\cdot O(k\varepsilon) + O(k^2\varepsilon^2\ln(k\varepsilon))\big) \\
&\quad -d_1\cdot O(k\varepsilon) + O(k^2\varepsilon^2\ln(k\varepsilon)).
\end{aligned} \tag{5.7}$$

Similarly, on Γ_ε^-

$$u_\varepsilon(x) = \int_{\Gamma_\varepsilon^-} g_\varepsilon^e(x,y)\frac{\partial u_\varepsilon(y)}{\partial\nu}ds_y.$$

It follows that

$$\begin{aligned}
u_\varepsilon(\varepsilon X, 0) &= -\int_0^1 G_\varepsilon^e(X,Y)\varepsilon\varphi_2(Y)dY \\
&= -\frac{1}{\pi}\left(\frac{\alpha}{p}-\frac{\alpha}{q}\right)\cdot\varepsilon\ln(k\varepsilon) - \frac{1}{\pi}\left(\frac{\alpha}{p}-\frac{\alpha}{q}\right)(\gamma_0+h(X))\cdot\varepsilon + f(X) \\
&\quad -\left(\frac{\ln(k\varepsilon)}{\pi}+\frac{\alpha}{p}+\frac{\alpha}{q}\right)\cdot\varepsilon\cdot\big(d_1\cdot O(k\varepsilon) + O(k^2\varepsilon^2\ln(k\varepsilon))\big) + O(k^2\varepsilon^2\ln(k\varepsilon)).
\end{aligned} \tag{5.8}$$

By applying Lemma 5.1, we can conclude for the field enhancement on the apertures as stated below.

THEOREM 5.6. *Let function h be defined by (5.6). The wave fields on the apertures Γ_ε^+ and Γ_ε^- are*

$$u_\varepsilon(x_1,1) = -\frac{2i}{\pi}\ln(k\varepsilon) - \frac{2i}{\pi}\left(\gamma_0 + h(x_1/\varepsilon)\right) + 2 + O(\varepsilon\ln^3\varepsilon)$$
$$u_\varepsilon(x_1,0) = -\frac{2i}{\pi}\ln(k\varepsilon) - \frac{2i}{\pi}\left(\gamma_0 + h(x_1/\varepsilon)\right) + O(\varepsilon\ln^3\varepsilon)$$

at the odd resonant frequencies $k = Re\, k_{n,1}$. At the even resonant frequencies $k = Re\, k_{n,2}$, the wave fields on the slit apertures are

$$u_\varepsilon(x_1,1) = -\frac{2i}{\pi}\ln(k\varepsilon) - \frac{2i}{\pi}\left(\gamma_0 + h(x_1/\varepsilon)\right) + 2 + O(\varepsilon\ln^3\varepsilon)$$
$$u_\varepsilon(x_1,0) = \frac{2i}{\pi}\ln(k\varepsilon) + \frac{2i}{\pi}\left(\gamma_0 + h(x_1/\varepsilon)\right) + O(\varepsilon\ln^3\varepsilon).$$

It is seen that the leading order of the resonant mode is a constant along the slit apertures with an order of $O\ln(k\varepsilon)$, and the enhancement due to the resonance is of order $O(1/\varepsilon)$.

6. Quantitative analysis of field enhancement in the non-resonant quasi-static regime.

In this section we consider the field enhancement in the quasi-static regime, i.e. when the wavenumber $k \ll 1$. Note that the resonance does not occur for the scattering problem (1.1) in this regime, as observed from Theorem 4.3. However, there is still strong enhancement of the electric field in the slit in the case when $\varepsilon \to 0$. This is proved rigorously in [19]. Here we derive the asymptotic expansion of the wave modes in the slit and over the slit apertures, and study the enhancement order in this regime. Note that no enhancement occurs for the scattered wave in the far field for such case, thus we do not elaborate here.

6.1. Field enhancement in the slit.

Based on the expansion (5.5) and Lemma 5.4, we can deduce that in the slit region S_ε^{int},

$$u_\varepsilon(x) = u_0(x_2) + u_\infty(x_1,x_2),$$

where

$$u_0(x_2) = \Big[\alpha + d_1\cdot O(k\varepsilon) + O(k^2\varepsilon^2\ln(k\varepsilon))\Big]\left[\frac{\cos(kx_2)}{k\sin k}\left(\frac{1}{p}+\frac{1}{q}\right) + \frac{\cos(k(1-x_2))}{k\sin k}\left(\frac{1}{p}-\frac{1}{q}\right)\right],$$

and

$$u_\infty(x_1,x_2) = \sum_{m\ge 1} a_m \cos\frac{m\pi x_1}{\varepsilon}\exp\left(-k_2^{(m)}x_2\right) + \sum_{m\ge 1} b_m\cos\frac{m\pi x_1}{\varepsilon}\exp\left(-k_2^{(m)}(1-x_2)\right).$$

If $k \ll 1$ and $\varepsilon \ll 1$, an expansion of (5.1) and (5.2) leads to

$$\frac{1}{p}\cdot\frac{1}{k\sin k} = \frac{1}{(\cos k+1)\alpha}\left(1 + O(k^2\varepsilon\ln(k\varepsilon))\right)$$

and

$$\frac{1}{q}\cdot\frac{1}{k\sin k} = \frac{1}{(\cos k-1)\alpha}\left(1 + O(\varepsilon\ln(k\varepsilon))\right).$$

Thus we have

$$u_0(x_2) = \left[1 + O(k^2\varepsilon\ln(k\varepsilon))\right] \frac{\cos(k(x_2 - 1/2))}{\cos(k/2)} + [1 + O(\varepsilon\ln(k\varepsilon))] \frac{\sin(k(x_2 - 1/2))}{\sin(k/2)}$$
$$+ d_1 \cdot O(k\varepsilon) + O(k^2\varepsilon^2\ln(k\varepsilon)).$$

Applying the Taylor expansion yields

$$\begin{aligned} u_0(x_2) &= \left[1 + O(k^2\varepsilon\ln(k\varepsilon))\right] \cdot \left(1 + O(k^2)\right) + [1 + O(\varepsilon\ln(k\varepsilon))] \cdot \left(2x_2 - 1 + O(k^2)\right) \\ &\quad + d_1 \cdot O(k\varepsilon) + O(k^2\varepsilon^2\ln(k\varepsilon)) \\ &= 2x_2 + O(k^2) + O(\varepsilon\ln(k\varepsilon)) + d_1 \cdot O(k\varepsilon). \end{aligned}$$

On the other hand, $u_\infty \sim O(\exp(-1/\varepsilon))$ as observed from Lemma 5.4.

From the above formulas, it is clear that there is no enhancement for the magnetic field u_ε if $k \ll 1$ and $\varepsilon \ll 1$. However, the transition of the magnetic field u_ε along the negative x_2 direction resembles a linear function with a slope of -2 in the slit. This is in contrast with the incident field, which changes with a rate of $O(k)$ in the slit. Such fast transition of magnetic field from the upper to lower slit aperture, compared to the incident wave, induces strong electric field enhancement as stated by the following theorem.

THEOREM 6.1. *If $k \ll 1$ and $\varepsilon \ll 1$, the electric field $E_\varepsilon = [E_{\varepsilon,1}, E_{\varepsilon,2}, 0]$ in S_ε^{int}, where*

$$E_{\varepsilon,1} = \frac{2i}{k\sqrt{\tau_0/\mu_0}} + O(\varepsilon\ln(k\varepsilon)/k) + O(k) + d_1 \cdot O(\varepsilon)$$

and

$$E_{\varepsilon,2} \sim O(\exp(-1/\varepsilon)/k).$$

τ_0 and μ_0 is the electric permittivity and magnetic permeability in the vacuum respectively.

Proof. Note that in the TM case, the magnetic field is given by

$$H_\varepsilon = [0, 0, u_\varepsilon].$$

Therefore, by the Ampere's law

$$\nabla \times H_\varepsilon = [\partial u_\varepsilon/\partial x_2, -\partial u_\varepsilon/\partial x_1, 0] = -i\omega\tau_0 E_\varepsilon,$$

we have

$$E_{\varepsilon,1} = \frac{2i}{k\sqrt{\tau_0/\mu_0}} + O(\varepsilon\ln(k\varepsilon)/k) + O(k) + d_1 \cdot O(\varepsilon)$$

and

$$E_{\varepsilon,2} = -\partial u_\infty/\partial x_1 \cdot i/\omega\tau_0 \sim O(\exp(-1/\varepsilon)/k).$$

□

If the thickness of the slab $\ell \neq 1$ and $\varepsilon \ll \ell$, then the electric field inside the slit $S_\varepsilon^{int} := \{x \in S_\varepsilon \mid x_2 \gg \varepsilon, \ell - x_2 \gg \varepsilon\}$ can be derived directly by a scaling argument.

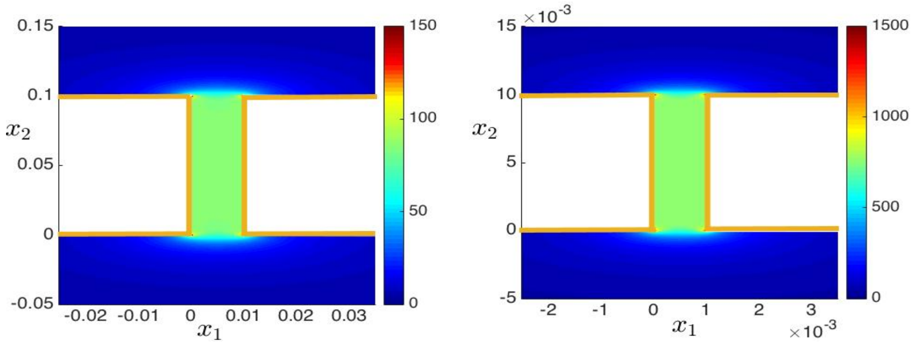

FIG. 6.1. *Left:* $|E_\varepsilon| / |E^{inc}|$ *for* $k = 0.1$, $\ell = 0.1$, *and* $\varepsilon = 0.01$; *Right:* $|E_\varepsilon| / |E^{inc}|$ *for* $k = 0.1$, $\ell = 0.01$, *and* $\varepsilon = 0.001$. *Here* E^{inc} *is the incident electric field.*

PROPOSITION 6.2. *If* $k\ell \ll 1$ *and* $\varepsilon \ll \min\{1, \ell\}$, *then the electric field is given by* $E_\varepsilon = [E_{\varepsilon,1}, E_{\varepsilon,2}, 0]$ *in* S_ε^{int}, *where*

$$E_{\varepsilon,1} = \frac{2i}{k\ell\sqrt{\tau_0/\mu_0}} + O(\varepsilon \ln(k\ell\varepsilon)/(k\ell)) + O(k\ell) + d_1 \cdot O(\varepsilon)$$

and

$$E_{\varepsilon,2} \sim O(\exp(-1/\varepsilon)/(k\ell)).$$

Therefore, an enhancement order of $O(1/(k\ell))$ is obtained for the electric field in the slit in the quasi-static regime. Moreover, the leading order of the electric field is a constant. This is demonstrated numerically in Figure 6.1, where $|E_\varepsilon| / |E^{inc}|$ is plotted for $k = 0.1$, $\ell = 0.1$, $\varepsilon = 0.01$ and $k = 0.1$, $\ell = 0.01$, $\varepsilon = 0.001$ respectively under the normal incidence.

6.2. Field enhancement on the slit apertures. Again, there is no enhancement for the magnetic field on the aperture, as observed from (5.7) and (5.8). Next we demonstrates the enhancement of the electric field. From Lemma 5.2, it is seen that

$$\begin{aligned} \frac{\partial u_\varepsilon}{\partial x_2}(x_1, 1) = &-K^{-1}1 \cdot \left(d_1 \cdot O(k) + \frac{\alpha}{p} + \frac{\alpha}{q} \right) \\ &+ \left(\frac{\alpha}{p} + \frac{\alpha}{q} \right) \big(d_1 \cdot O(k\varepsilon) + O(k^2\varepsilon^2 \ln(k\varepsilon)) \big) + O(k^2\varepsilon \ln(k\varepsilon)), \end{aligned} \tag{6.1}$$

$$\begin{aligned} \frac{\partial u_\varepsilon}{\partial x_2}(x_1, 0) = &K^{-1}1 \cdot \left(d_1 \cdot O(k) + \frac{\alpha}{p} - \frac{\alpha}{q} \right) \\ &+ \left(\frac{\alpha}{p} + \frac{\alpha}{q} \right) \big(d_1 \cdot O(k\varepsilon) + O(k^2\varepsilon^2 \ln(k\varepsilon)) \big) + O(k^2\varepsilon^2 \ln(k\varepsilon)). \end{aligned} \tag{6.2}$$

If $k \ll 1$ and $\varepsilon \ll 1$, then

$$\frac{1}{p} = \frac{k \sin k}{(\cos k + 1)\alpha}(1 + O(\varepsilon \ln(k\varepsilon))) = O(k^2), \tag{6.3}$$

$$\frac{1}{q} = \frac{k \sin k}{(\cos k - 1)\alpha}(1 + O(\varepsilon \ln(k\varepsilon))) = -\frac{2}{\alpha}\left(1 + O(k^2)\right)(1 + O(\varepsilon \ln(k\varepsilon))). \tag{6.4}$$

Substituting into (6.1) and (6.2), we have

$$\frac{\partial u_\varepsilon}{\partial x_2}(x_1,1)=\frac{\partial u_\varepsilon}{\partial x_2}(x_1,0)=2K^{-1}1+O(\varepsilon\ln(k\varepsilon))+d_1\cdot O(k)+d_1\cdot O(k\varepsilon). \tag{6.5}$$

On the other hand, from (5.7) and (5.8), it is clear that

$$\frac{\partial u_\varepsilon}{\partial x_1}(x_1,1)=-\frac{1}{\pi}\left(\frac{\alpha}{p}+\frac{\alpha}{q}\right)\cdot\frac{1}{\varepsilon}h'(X)\cdot\varepsilon+d_1\cdot O(k\varepsilon)+O(k^2\varepsilon^2\ln(k\varepsilon)),$$
$$\frac{\partial u_\varepsilon}{\partial x_1}(x_1,0)=-\frac{1}{\pi}\left(\frac{\alpha}{p}-\frac{\alpha}{q}\right)\cdot\frac{1}{\varepsilon}h'(X)\cdot\varepsilon+d_1\cdot O(k\varepsilon^2\ln(k\varepsilon))+O(k^2\varepsilon^2\ln(k\varepsilon)),$$

where $h(X)$ is defined by (5.6). By substituting (6.3) and (6.4), we obtain

$$\frac{\partial u_\varepsilon}{\partial x_1}(x_1,1)=\frac{2}{\pi}h'(X)+O(\varepsilon\ln(k\varepsilon))+d_1\cdot O(k\varepsilon), \tag{6.6}$$
$$\frac{\partial u_\varepsilon}{\partial x_1}(x_1,0)=-\frac{2}{\pi}h'(X)+O(\varepsilon\ln(k\varepsilon))+d_1\cdot O(k\varepsilon^2\ln(k\varepsilon)). \tag{6.7}$$

In light of (6.5) - (6.7), a combination of the Ampere's law and a scaling argument leads to the following Theorem.

THEOREM 6.3. *If $k\ell\ll 1$ and $\varepsilon\ll 1$, then the electric field*

$$E_\varepsilon(x_1,\ell)=\frac{2i}{k\ell\sqrt{\tau_0/\mu_0}}[K^{-1}1,-h'(X)/\pi,0]+d_1\cdot O(1)+O(\varepsilon\ln(k\ell\varepsilon)/(k\ell))+d_1\cdot O(\varepsilon)$$

and

$$E_\varepsilon(x_1,0)=\frac{2i}{k\ell\sqrt{\tau_0/\mu_0}}[K^{-1}1,h'(X)/\pi,0]+d_1\cdot O(1)+O(\varepsilon\ln(k\ell\varepsilon)/(k\ell))+d_1\cdot O(\varepsilon)$$

on the upper and lower gap apertures respectively.

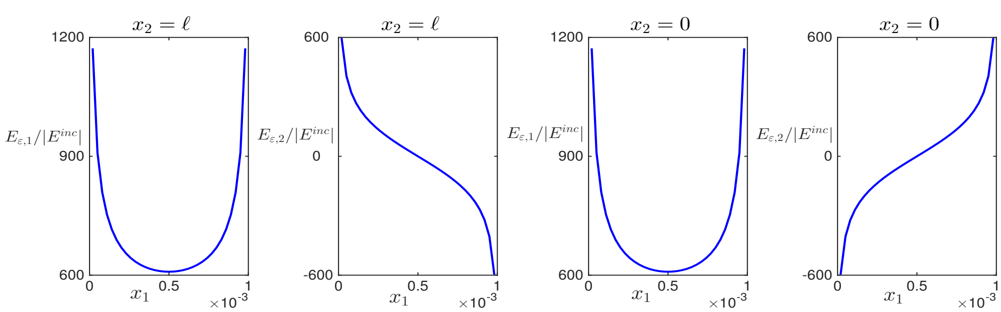

FIG. 6.2. *$E_{\varepsilon,1}/\left|E^{inc}\right|$ and $E_{\varepsilon,2}/\left|E^{inc}\right|$ on the upper and lower gap apertures when $k=0.1$, $\ell=0.01$, and $\varepsilon=0.001$. E^{inc} is the incident electric field. It is seen that $E_{\varepsilon,1}(x_1,\ell)=E_{\varepsilon,1}(x_1,0)$ and $E_{\varepsilon,2}(x_1,\ell)=-E_{\varepsilon,2}(x_1,0)$. This is consistent with the asymptotic expansions in Theorem 6.3.*

Again, an enhancement order of $O\left(1/(k\ell)\right)$ is obtained for the electric field on the slit apertures in the quasi-static regime. Figure 6.2 illustrates the enhancement of the electric field when $k=0.1$, $\ell=0.01$, and $\varepsilon=0.001$ and under the normal incidence.

7. Conclusion. This paper gives a complete picture of the enhancement mechanism for the scattering of a narrow slit and demonstrates that the field enhancement can be induced by either Fabry-Perot type scattering resonances or certain non-resonant effect in the quasi-static regime. The asymptotic expansions of the resonances were rigorously derived and the scattering enhancement was analyzed quantitatively at both resonant frequencies and non-resonant frequencies in the quasi-static regime. The study of the single slit sheds some light for the field enhancement of other subwavelength structures such as an array of narrow slits and slits with real metals. It is expected that similar enhancement mechanisms will also occur for these two configurations. In addition, other enhancement mechanisms, including surface plasmonic resonances and spoof surface plasmonic resonances, will be present. This is being explored and will be reported elsewhere.

REFERENCES

[1] M. Abramowitz and I. Stegun, eds., *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, NBS Appl. Math. Ser. 55, National Bureau of Standards, Washington, DC, 1964.

[2] R. Adams and J. Fournier, *Sobolev Spaces*, vol. 140, Academic Press, 2003.

[3] H. Ammari, H. Kang, and H. Lee, *Layer Potential Techniques in Spectral Analysis*, Mathematical Surveys and Monographs, Vol. 153, Amer. Math. Soc., Rhode Island, 2009.

[4] H. Ammari and H. Zhang, *A mathematical theory of super-resolution by using a system of sub-wavelength Helmholtz resonators*, Comm. Math. Phys., **337** (2015), 379-428.

[5] H. Ammari, B. Fitzpatrick, D. Gontier, H. Lee and H. Zhang, *Minnaert resonances for acoustic waves in bubbly media*, submitted.

[6] S. Astilean, Ph. Lalanne and M. Palamaru, *Light transmission through metallic channels much smaller than the wavelength*, Opt. Commun., **175** (2000), 265-273.

[7] E. Bonnetier and F. Triki, *Asymptotic of the Green function for the diffraction by a perfectly conducting plane perturbed by a sub-wavelength rectangular cavity*, Math. Meth. Appl. Sci., **33** (2010), 772-798.

[8] J. F. Babadjian, E. Bonnetier and F. Triki, *Enhancement of electromagnetic fields caused by interacting subwavelength cavities*, Multiscale Model. Simul., **8** (2010), 1383-1418.

[9] X. Chen *et al*, *Atomic layer lithography of wafer-scale nanogap arrays for extreme confinement of electro-magnetic waves*, Nat. Commun., **4** (2013), 2361.

[10] M. Clausel, M. Durufle, P. Joly, and S. Tordeux, *A mathematical analysis of the resonance of the finite thin slots*, Appl. Numer. Math., **56** (2006), 1432-1449.

[11] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, Applied Mathematical Sciences, vol. 93, Springer-Verlag, Berlin, 1998.

[12] T. W. Ebbesen, H. J. Lezec, H. F. Ghaemi, T. Thio, and P. A. Wolff, *Extraordinary optical transmission through sub-wavelength hole arrays*, Nature, **391** (1998), 667-669.

[13] F. J. Garcia-Vidal, L. Martin-Moreno, T. W. Ebbesen, and L. Kuipers, *Light passing through subwavelength apertures*, Rev. Modern Phys., **82** (2010), 729-787.

[14] I. Gradshteyn and I. Ryzhik, Table of Integrals, Series, and Products. Academic Press, 2014.

[15] R. Kress, *Linear Integral Equations*, Applied Mathematical Sciences, vol. 82, Springer-Verlag, Berlin, 1999.

[16] G. A. Kriegsmann, *Complete transmission through a two-dimensional diffraction grating*, SIAM J. Appl. Math., **65** (2004), 24-42.

[17] L. Lewin, *Theory of Waveguides: Techniques for the Solution of Waveguide Problems*, New York, Halsted Press, 1975.

[18] J. Lin, S.-H. Oh, H.-M. Nguyen, and F. Reitich, *Field enhancement and saturation of millimeter waves inside a metallic nanogap*, Opt. Express, **22** (2014), pp. 14402-14410.

[19] J. Lin and F. Reitich, *Electromagnetic field enhancement in small gaps: a rigorous mathematical theory*, SIAM J. Appl. Math., **75** (2015), 2290-2310.

[20] P. Joly and S. Tordeux, *Asymptotic analysis of an approximate model for time harmonic waves in media with thin slots*, ESAIM Math. Model. Numer. Anal., **40** (2006), 63-97.

[21] P. Joly and S. Tordeux, *Matching of asymptotic expansions for wave propagation in media with thin slots I: The asymptotic expansion*, Multiscale Model. Simul., **5** (2006), 304-336.

[22] P. Joly and S. Tordeux, *Matching of asymptotic expansions for wave propagation in media*

with thin slots II: the error estimates, ESAIM: M2AN, **42** (2008) 193-221.

[23] M. A. Seo *et al.*, *Terahertz field enhancement by a metallic nano slit operating beyond the skin-depth limit*, Nat. Photonics, **3** (2009), 152-156.

[24] B. Sturman, E. Podivilov, and M. Gorkunov, *Transmission and diffraction properties of a narrow slit in a perfect metal*, Phys. Rev. B, **82** (2010), 115419.

[25] Y. Takakura, *Optical resonance in a narrow slit in a thick metallic screen*, Phys. Rev. Lett., **99** (2001), 5601-5603.

[26] F. Yang and J. R. Sambles, *Resonant transmission of microwaves through a narrow metallic slit*, Phys. Rev. Lett., **89** (2002), 063901.