CHAPTER 4

Eigenvectors and Eigenvalues

In this Chapter we will learn the important notion of eigenvectors and eigenvalues of matrices and linear transformation in general.

4.1 Eigenvectors

**Definition 4.1.** An eigenvector of a linear transformation $T \in L(V,V)$ is a nonzero vector $u \in V$ such that $Tu = \lambda u$ for some scalar $\lambda \in K$.

$\lambda$ is called the eigenvalue of the eigenvector $u$.

The space $V_\lambda := \{u : Tu = \lambda u\} \subset V$ is called the eigenspace of the eigenvalue $\lambda$.

We have the same definition if $T \in L(\mathbb{R}^n, \mathbb{R}^n)$ is represented by an $n \times n$ matrix $A$.

**Note.** $V_\lambda$ contains $0$, although $0$ is not an eigenvector by definition.

**Properties 4.2.** The eigenspace $V_\lambda = \text{Nul}(A - \lambda \text{Id})$ is a vector space.

In particular, any linear combinations of eigenvectors with eigenvalue $\lambda$ is again an eigenvector with eigenvalue $\lambda$ if it is nonzero.

**Properties 4.3.** $\lambda$ is an eigenvalue of $A$ if and only if $\det(A - \lambda \text{Id}) = 0$. 
Chapter 4. Eigenvectors and Eigenvalues 4.1. Eigenvectors

General strategy:

**Step 1.** Find the eigenvalues $\lambda$ using determinant.

**Step 2.** For each eigenvalue $\lambda$, find the eigenvectors by solving the linear equations $(\mathbf{A} - \lambda \mathbf{I}) \mathbf{x} = \mathbf{0}$, which have non-trivial solutions.

**Example 4.1.** Let \( \mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \). To find eigenvalues,

\[
\det \begin{pmatrix} 1 - \lambda & 1 \\ 4 & 1 - \lambda \end{pmatrix} = (1 - \lambda)(1 - \lambda) - 4 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0,
\]
hence $\lambda = 3$ or $\lambda = -1$.

For $\lambda = 3$, we have \( \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0} \implies \begin{pmatrix} t \\ 2t \end{pmatrix} \) are eigenvectors for all $t \in \mathbb{R}$.

For $\lambda = -1$, we have \( \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0} \implies \begin{pmatrix} t \\ -2t \end{pmatrix} \) are eigenvectors for all $t \in \mathbb{R}$.

Some useful theorems:

**Theorem 4.4.** The eigenvalues of a triangular matrix (in particular diagonal matrix) are the entries on its main diagonal.

**Theorem 4.5** (Invertible Matrix Theorem). $\mathbf{A}$ is invertible if and only if 0 is not an eigenvalue of $\mathbf{A}$.

**Example 4.2.** $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ has eigenvalues $\lambda = 1, 2, 3$.

**Example 4.3.** $\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ has eigenvalues $\lambda = 1, 3$ only. $V_3$ is 1-dimensional spanned by $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, while $V_1$ is 2-dimensional spanned by $\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \}$.

**Example 4.4.** $\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ has eigenvalues $\lambda = 1, 3$ only. $V_3$ is 1-dimensional spanned by $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, but $V_1$ is only 1-dimensional spanned by $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. 

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Example 4.5. Existence of eigenvalues depend on the field $K$. For example, $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has no eigenvalues in $\mathbb{R}$, but it has two complex eigenvalues $\lambda = i$ with eigenvectors $\begin{pmatrix} i \\ 1 \end{pmatrix}$, and $\lambda = -i$ with eigenvectors $\begin{pmatrix} -i \\ 1 \end{pmatrix}$.

Theorem 4.6. If $v_1, \ldots, v_r$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$, then the set $\{v_1, \ldots, v_r\}$ is linearly independent.

4.2 Determinants

To find eigenvalues, we need to solve the characteristic equation

$$\det(A - \lambda I) = 0$$

Let us recall the definition of Determinants of a matrix.

Definition 4.7. The determinant is the unique function $\det : M_{n \times n} \rightarrow K$ such that it satisfies the following properties:

- Determinant of identity matrix is 1: $\det(I) = 1$
- It is skew-symmetric: Interchange two rows gives a sign:

$$\det \begin{pmatrix} \vdots & -r_i & - \\ -r_j & \vdots \end{pmatrix} = -\det \begin{pmatrix} \vdots & -r_j & - \\ -r_i & \vdots \end{pmatrix}$$

In particular, if the matrix has two same rows, $\det = 0$.
- It is multilinear, i.e. it is linear respective to rows:
  - Addition:

$$\det \begin{pmatrix} \vdots & -r + s & - \\ \vdots \end{pmatrix} = \det \begin{pmatrix} \vdots & -r & - \\ \vdots \end{pmatrix} + \det \begin{pmatrix} \vdots & -s & - \\ \vdots \end{pmatrix}$$
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- Scalar multiplication: for \( k \in K \),

\[
\det \left( \begin{array}{ccc}
\vdots & \vdots & \vdots \\
-k \cdot \mathbf{r} & \cdots & \cdots \\
\vdots & \vdots & \vdots \\
\end{array} \right) = k \det \left( \begin{array}{ccc}
\vdots & \vdots & \vdots \\
\mathbf{r} & \cdots & \cdots \\
\vdots & \vdots & \vdots \\
\end{array} \right)
\]

- zero:

\[
\det \left( \begin{array}{ccc}
\vdots & 0 & \vdots \\
\vdots & \vdots & \vdots \\
\end{array} \right) = 0
\]

**Remark.** The definition of the determinant means that \( \det \mathbf{A} \) is the *signed volume* of the “parallelepiped” spanned by the columns of \( \mathbf{A} \) (and \( |\det \mathbf{A}| \) is the volume).

From this definition, one can calculate the determinant of \( \mathbf{A} \) using the *reduced row echelon form*, because it says that row operations do not change the determinant:

\[
\det \left( \begin{array}{ccc}
\vdots & \mathbf{r}_i + c \mathbf{r}_j & \vdots \\
\vdots & \mathbf{r}_j & \vdots \\
\end{array} \right) = \det \left( \begin{array}{ccc}
\vdots & \mathbf{r}_i & \vdots \\
\vdots & \mathbf{r}_j & \vdots \\
\end{array} \right) + c \det \left( \begin{array}{ccc}
\vdots & \mathbf{r}_j & \vdots \\
\vdots & \mathbf{r}_j & \vdots \\
\end{array} \right) = \det \left( \begin{array}{ccc}
\vdots & \mathbf{r}_i & \vdots \\
\vdots & \mathbf{r}_j & \vdots \\
\end{array} \right)
\]

This also implies that

**Properties 4.8.** If \( \mathbf{A} \) is triangular (in particular diagonal), then \( \det \mathbf{A} \) is the product of the entries on the main diagonal of \( \mathbf{A} \).

**Example 4.6.**

\[
\det \left( \begin{array}{ccc}
0 & 2 & 3 \\
1 & 0 & 1 \\
0 & 3 & 2 \\
\end{array} \right) R_1 \leftrightarrow R_2 = \det \left( \begin{array}{ccc}
1 & 0 & 1 \\
0 & 2 & 3 \\
0 & 3 & 2 \\
\end{array} \right)
\]

\[
= -2 \det \left( \begin{array}{ccc}
1 & 0 & 1 \\
0 & 1 & 3/2 \\
0 & 3 & 2 \\
\end{array} \right)
\]

\[
= -2 \cdot 1 \cdot 1 \cdot (3/2) = -3.
\]

\[
\frac{1}{2} R_2 = -2 \det \left( \begin{array}{ccc}
1 & 0 & 1 \\
0 & 1 & 3/2 \\
0 & 3 & 2 \\
\end{array} \right)
\]

\[
R_3 = 3 R_2 = -2 \det \left( \begin{array}{ccc}
1 & 0 & 1 \\
0 & 1 & 3/2 \\
0 & 0 & -5/2 \\
\end{array} \right)
\]

\[
= -2 \cdot 1 \cdot 1 \cdot (-5/2) = 5.
\]
Properties 4.9. The properties of determinant under row operations means that \(\det\) change as follows if you multiply your matrix by the \textbf{elementary matrices} on the left:

- \(E = \begin{pmatrix}
  \cdots \\
  0 & 1 \\
  1 & 0 \\
  \cdots
\end{pmatrix}^i_j\) : Interchanging two rows:
  \[
  \det(EA) = -\det(A)
  \]

- \(E = \begin{pmatrix}
  \cdots \\
  k \\
  \cdots
\end{pmatrix}^i_i\) : Scalar multiplying \(i\)-th row by \(k\):
  \[
  \det(EA) = k\det(A)
  \]

- \(E = \begin{pmatrix}
  \cdots \\
  1 & c \\
  0 & 1 \\
  \cdots
\end{pmatrix}^i_j\) : Adding multiples of \(j\)-th row to \(i\)-th row:
  \[
  \det(EA) = \det(A)
  \]

Here the \(\cdots\) means it is 1 on the diagonal and 0 otherwise outside the part shown.

The determinant has the following very useful properties

\textbf{Theorem 4.10.} Let \(A, B\) be \(n \times n\) matrices

- \(A\) is invertible if and only if \(\det A \neq 0\)
- \(\det AB = (\det A)(\det B)\)
- \(\det A^T = \det A\)
- \(\det kA = k^n \det A\)
Corollary 4.11. If $A$ is invertible, $\det(A^{-1}) = \det(A)^{-1}$.

If $Q$ is orthogonal matrix, then $\det(Q) = \pm 1$

Alternative way to compute determinant is

**Theorem 4.12 (The Laplace Expansion Theorem).** If $A$ is $n \times n$ matrix, $\det A$ is computed by

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

or

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

where $C_{ij} = (-1)^{i+j} \det A_{ij}$ and $A_{ij}$ is the submatrix obtained by deleting the $i$-th row and $j$-th column.

**Example 4.7.** Using the same example above, using the first row,

$$\det \begin{pmatrix} 0 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 3 & 2 \end{pmatrix} = 0 \cdot \det \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix} - 2 \cdot \det \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} + 3 \cdot \det \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = 0 - 2 \cdot 2 + 3 \cdot 3 = 5$$

or using the second column,

$$\det \begin{pmatrix} 0 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 3 & 2 \end{pmatrix} = -2 \cdot \det \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} + 0 \cdot \det \begin{pmatrix} 0 & 3 \\ 0 & 2 \end{pmatrix} - 3 \cdot \det \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix} = -2 \cdot 2 + 0 - 3 \cdot (-3) = 5$$

Of course, the smart way is to choose the first column, because there are 2 zeros, making computation easier :)

### 4.3 Characteristic polynomial

**Definition 4.13.** If $A$ is $n \times n$ matrix,

$$p(\lambda) := \det(A - \lambda \text{Id})$$

is a polynomial in $\lambda$ of degree $n$, called the **characteristic polynomial**.

Therefore eigenvalues are the roots of the characteristic equations $p(\lambda) = 0$. 

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Definition 4.14. We define the notion of multiplicity:

- The dimension of the eigenspace $V_\lambda$ is called the **geometric multiplicity**.
- The multiplicity of the root $\lambda$ of $p(\lambda) = 0$ is called the **algebraic multiplicity**.

From the fundamental theorem of algebra, we note that since any polynomial of degree $n$ has $n$ roots (with repeats),

The algebraic multiplicities add up to $n$

**Example 4.8.** If $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, then the characteristic polynomial is

$$p(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix} 3 - \lambda & 0 & 0 \\ 0 & 1 - \lambda & 1 \\ 0 & 0 & 1 - \lambda \end{pmatrix} = (1 - \lambda)^2(3 - \lambda)$$

Note that $\lambda = 1$ is a multiple root, hence the algebraic multiplicity of $\lambda = 1$ is 2.

On the other hand, the eigenspace $V_1$ is spanned by $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ only (see Exercise 4.4), so $\lambda = 1$ has geometric multiplicity = 1 only.

In other words:

**Geometric multiplicity is in general not the same as algebraic multiplicity.**

**Theorem 4.15.** The characteristic polynomial $p(\lambda)$ has the following properties:

- The top term is $\lambda^n$ with coefficient $(-1)^n$
- The coefficient of $\lambda^{n-1}$ is $(-1)^{n-1}\text{Tr} A$
- The constant term is $\det A$. 

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Finally, we have the following interesting result, which can be used to calculate inverse of matrix!

**Theorem 4.16** (Cayley-Hamilton Theorem). If \( p(\lambda) \) is the characteristic polynomial of \( A \), then

\[
p(A) = O
\]

where \( O \) is the zero matrix.

**Example 4.9** (cont’d). Since \((1 - \lambda)^2(3 - \lambda) = 3 - 7\lambda + 5\lambda^2 - \lambda^3\), the matrix

\[
A = \begin{pmatrix}
3 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
\]

satisfies

\[
3\text{Id} - 7A + 5A^2 - A^3 = O
\]

Therefore

\[
3\text{Id} = 7A - 5A^2 + A^3
\]

Multiplying both sides by \( A^{-1} \) we obtain

\[
3A^{-1} = 7\text{Id} - 5A + A^2
\]

This gives the inverse of \( A \) easily by usual matrix multiplication only.

### 4.4 Similarity

**Definition 4.17.** If \( A, B \) are \( n \times n \) matrices, then \( A \) is similar to \( B \) if there is an invertible matrix \( P \) such that

\[
A = PB\text{P}^{-1}
\]

Since if \( Q = P^{-1} \), then also \( B = QAQ^{-1} \) and \( B \) is similar to \( A \). Therefore we can just say \( A \) and \( B \) are similar. We usually write

\[
A \sim B
\]

We have the following properties:

**Theorem 4.18.** If \( A \) and \( B \) are similar, then

- They have the same determinant
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- They have the same characteristic polynomial
- They have the same eigenvalues
- They have the same algebraic and geometric multiplicities.

The polynomials of symmetric matrices are also related, which is very useful if the matrix is similar to a diagonal matrix (see next Chapter)

**Theorem 4.19.** If \( A = PBP^{-1} \)

then for any integer \( n \),

\[ A^n = PB^nP^{-1} \]

In particular, for any polynomial \( p(x) \),

\[ p(A) = Pp(B)P^{-1} \]

Recall that for a linear transformation \( T \in L(V, V) \), it can be represented as an \( n \times n \) matrix \( \mathbb{R}^n \rightarrow \mathbb{R}^n \) with respect to a basis \( B \) of \( V \):

\[ [T]_B : [x]_B \mapsto [Tx]_B \]

Also recall the change of basis matrix

\[ P^B_{B'} : [x]_B \mapsto [x]_{B'} \]

Therefore we have the following interpretation of similar matrix:

**Theorem 4.20.** Let \( T \in L(V, V) \) such that \([T]_B = A\), \([T]_{B'} = B\) and \( P = P^B_{B'}\).

Then \( A \) is similar to \( B \):

\[ A = P^{-1}BP \]

In other words, similar matrix represents the same linear transformation with respect to different basis!

Pictorially, we have

\[
\begin{array}{cccc}
[x]_B & \xrightarrow{A} & [Tx]_B \\
\downarrow{P} & & \uparrow{P^{-1}} \\
[x]_{B'} & \xrightarrow{B} & [Tx]_{B'}
\end{array}
\]
This means that for the linear transformation $T : V \rightarrow V$, if we choose a “nice basis” $B$ of $B$, the matrix $[T]_B$ can be very nice! One choice of “nice basis” is given by diagonalization, which means under this basis, the linear transformation is represented by a diagonal matrix. We will study this in more detail in the next Chapter.

**Example 4.10.** In $\mathbb{R}^2$, let $\mathcal{E} = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$ be the standard basis, and $B = \{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix}\}$ be another basis. Let $T$ be the linear transformation represented in the standard basis $\mathcal{E}$ by $A = \begin{pmatrix} 14/5 & 2/5 \\ 2/5 & 11/5 \end{pmatrix}$. Then we can diagonalize $A$ as follows:

$$\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 14/5 & 2/5 \\ 2/5 & 11/5 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

See the picture below for the geometric meaning.