3. Two-Person Zero-Sum Games

3.1 Strategic Form.

The simplest mathematical description of a game is the strategic form. For a two-person zero-sum game, the payoff function of Player II (Column Chooser) is the negative of the payoff of Player I (Row Chooser), so we may restrict attention to the single payoff function of Player I.
**Definition:** The strategic form, or normal form, of a two-person zero-sum game is given by a triplet \((X, Y, A)\), where

1. \(X\) is a nonempty set, the set of strategies of Player I
2. \(Y\) is a nonempty set, the set of strategies of Player II
3. \(A\) is a real-valued function defined on \(X \times Y\). (Thus, \(A(x, y)\) is a real number for every \(x \in X\) and every \(y \in Y\).)

The interpretation is as follows. Simultaneously, Player I chooses \(x \in X\) and Player II chooses \(y \in Y\), each unaware of the choice of the other.

Then their choices are made known and I wins the amount \(A(x, y)\) from II.
If $A$ is negative, I pays the absolute value of this amount to II.

Thus, $A(x, y)$ represents the winnings of I and the losses of II.
Matrix Games

A finite two-person zero-sum game in strategic form, \((X, Y, A)\), is sometimes called a matrix game because the payoff function \(A\) can be represented by a matrix.

If \(X = \{x_1, \ldots, x_m\}\) and \(Y = \{y_1, \ldots, y_n\}\), then by the game matrix or payoff matrix we mean the matrix
\[ A = \begin{pmatrix}
  a_{11} & \cdots & a_{1n} \\
  \vdots & & \vdots \\
  a_{m1} & \cdots & a_{mn}
\end{pmatrix} \]

where \( a_{ij} = A(x_i, y_j) \),

In this form, Player I chooses a row, Player II chooses a column, and II pays I the entry in the chosen row and column.

Note that the entries of the matrix are the winnings of the row chooser and losses of the column chooser.
<table>
<thead>
<tr>
<th></th>
<th>$y_1$</th>
<th>$\cdots$</th>
<th>$y_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$a_{11}$</td>
<td></td>
<td>$a_{1n}$</td>
</tr>
<tr>
<td>\vdots</td>
<td>$\vdots$</td>
<td></td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$x_m$</td>
<td>$a_{m1}$</td>
<td></td>
<td>$a_{mn}$</td>
</tr>
</tbody>
</table>
3.2 Examples:

**Paper-Scissors-Rock**

Players I and II simultaneously display one of the three objects: paper (P), scissors (S), or rock (R). If they both choose the same object to display, there is no payoff. If they choose different objects, then scissors win over paper (scissors cut paper), rock wins over scissors (rock breaks scissors), and paper wins over rock (paper covers rock). If the payoff upon winning or losing is one unit, then the matrix of the game is as follows.
<table>
<thead>
<tr>
<th></th>
<th>$P$</th>
<th>$S$</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P$</td>
<td>0</td>
<td>$-1$</td>
<td>1</td>
</tr>
<tr>
<td>$S$</td>
<td>1</td>
<td>0</td>
<td>$-1$</td>
</tr>
<tr>
<td>$R$</td>
<td>$-1$</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
Matching Pennies:

Two players simultaneously choose heads or tails. Player I wins if the choices match and Player II wins otherwise. If the payoff upon winning or losing is one unit, then the payoff matrix of the game is as follow:

\[
\begin{array}{c|cc}
 & H & T \\
\hline
H & 1 & -1 \\
T & -1 & 1 \\
\end{array}
\]
Odd or Even

Players I and II simultaneously call out one of the numbers one or two. Player I’s name is Odd; he wins if the sum of the numbers is odd. Player II’s name is Even; she wins if the sum of the numbers is even. The amount paid to the winner by the loser is always the sum of the numbers in dollars. To put this game in strategic form we must specify X, Y and A. Here we may choose X = \{1, 2\}, Y = \{1, 2\}, and A as given in the following table.

\[ A(x, y) = \text{I’s winnings} = \text{II’s losses}. \]

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>-4</td>
</tr>
</tbody>
</table>


**Question:** How will the game play out?

This is not an easy question to answer. However, it is clear that there is no room for cooperation for the players. This is a typical example of competition.

The first principle that we can agree on is to simplify the game by removing dominated strategies.
Removing Dominated Strategies.

Definition. We say the ith row of a matrix $A = (a_{ij})$ dominates the kth row if $a_{ij} \geq a_{kj}$ for all $j$. We say the ith row of $A$ strictly dominates the kth row if $a_{ij} > a_{kj}$ for all $j$. Similarly, the jth column of $A$ dominates (strictly dominates) the kth column if $a_{ij} \leq a_{ik}$ (resp. $a_{ij} < a_{ik}$) for all $i$.

Anything Player I can achieve using a dominated row can be achieved at least as well using the row that dominates it. Hence dominated rows may be deleted from the matrix. A similar argument shows that dominated columns may be removed.

We may iterate this procedure and successively remove several rows and columns. (Examples to be given later)
Example: **Battle of Bismarck Sea**

In the critical stages of the struggle for New Guinea, intelligence reports indicated that the Japanese would move a troop and supply convoy from the port at the eastern tip of New Britain to Lae, which lies just west of New Britain or New Guinea. It could travel north of New Britain, where poor visibility was almost certain, or south of the Island, where the weather would be clear; in either case, the trip would take three days. General Kenney had the choice of concentrating the bulk of his reconnaissance aircraft on one route or the other. Once sighted, the convoy could be bombed until its arrival at Lae. In days of bombing time, Kenney’s staff estimated the following outcomes for the various choices:

For this game the second column is dominated by the first column. The Japanese will remove the second column from his consideration. Kenney, knowing the Japanese’s removal of the second column, will then play the first row. Therefore, \(<N, N>\) is the outcome of this game.

\[
\begin{array}{c|cc}
N & N & S \\
\hline
N & 2 & 2 \\
S & 1 & 3 \\
\end{array}
\]
MILITARY DECISION AND GAME THEORY

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The United States military doctrine of decision prescribes that a commander select the course of action which offers the greatest promise of success in view of the enemy's capabilities of opposing him. This paper analyzes two battle decisions of World War II, and develops the analogy between existing military doctrine and the 'theory of games' proposed by von Neumann. Current U. S. doctrine is conservative. The techniques of game theory permit analysis of the risk involved if the commander deviates from current doctrine to base his decision on his estimate of what his enemy intends to do rather than on what his enemy is capable of doing. The idea of 'mixed strategies' presents more difficulties but may be useful, particularly for command decisions for small military organizations.

VON NEUMANN and Morgenstern point out that in the early stages of the development of a new theory, application serves to corroborate theory. The theory of games has been most fully developed for the two-person situation, the conflict of two opposing individuals or groups. Almost all battle decisions involve two opposing military forces. Moreover, the student of game theory need not analyze numerous battles to learn the military philosophy of decision. The doctrine has been formalized and is available in military texts.

MILITARY-DECISION DOCTRINE

A military commander may approach decision with either of two philosophies. He may select his course of action on the basis of his estimate of what his enemy is able to do to oppose him. Or, he may make his selection on the basis of his estimate of what his enemy is going to do. The former is a doctrine of decision based on enemy capabilities; the latter, on enemy intentions.

The doctrine of decision of the armed forces of the United States is a doctrine based on enemy capabilities. A commander is enjoined to select

*The basic concepts of this paper were developed by the author, then a colonel in the U. S. Air Force, while a student at the Air War College, 1949-1950.
the course of action which offers the greatest promise of success in view of the enemy capabilities. The process of decision, as approved by the Joint Chiefs of Staff and taught in all service schools, is formalized in a five-step analysis called the Estimate of the Situation. These steps are illustrated in the following analysis of an actual World War II battle situation.

**BISMARCK SEA**

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**THE RABAUL-LAE CONVOY SITUATION**

General Kenney was Commander of the Allied Air Forces in the Southwest Pacific Area. The struggle for New Guinea reached a critical stage in February 1943. Intelligence reports indicated a Japanese troop and supply convoy was assembling at Rabaul (see Fig. 1). Lae was expected to be the unloading point. With this general background Kenney proceeded to make his five-step Estimate of the Situation.

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**Step 1. The Mission**

General MacArthur as Supreme Commander had ordered Kenney to intercept and inflict maximum destruction on the convoy. This then was Kenney's mission.

**Step 2. Situation and Courses of Action**

The situation as outlined above was generally known. The new critical factor was pointed out by Kenney's staff. Rain and poor visibility were predicted for the area north of New Britain. Visibility south of the island would be good.

The Japanese commander had two choices for routing his convoy from Rabaul to Lae. He could sail north of New Britain, or he could go south of that island. Either route required three days.

Kenney considered two courses of action, as he discusses in his memoirs. He could concentrate most of his reconnaissance aircraft either along the northern route where visibility would be poor, or along the southern route where clear weather was predicted. Mobility being one of the great advantages of air power, his bombing force could strike the convoy on either route once it was spotted.

**Step 3. Analysis of the Opposing Courses of Action**

With each commander having two alternative courses of action, four possible conflicts could ensue. These conflicts are pictured in Fig. 2.

**Step 4. Comparison of Available Courses of Action**

If Kenney concentrated on the northern route, he ensured one of the two battles of the top row of sketches. However, he alone could not determine which one of these two battles in the top row would result from his decision. Similarly, if Kenney concentrated on the southern route, he ensured one of the battles of the lower row. In the same manner, the Japanese commander could not select a particular battle, but could by his decision assure that the battle would be one of those pictured in the left column or one of those in the right column.

Kenney sought a battle which would provide the maximum opportunity for bombing the convoy. The Japanese commander desired the minimum exposure to bombing. But neither commander could determine the battle which would result from his own decision. Each commander had full and independent freedom to select either one of his alternative strategies. He had to do so with full realization of his opponent's freedom of choice. The particular battle which resulted would be determined by the two independent decisions.
The U.S. doctrine of decision—the doctrine that a commander base his action on his estimate of what the enemy is capable of doing to oppose him—dictated that Kenney select the course of action which offered the greatest promise of success in view of all of the enemy capabilities. If Kenney concentrated his reconnaissance on the northern route, he could expect two days of bombing regardless of his enemy's decision. If Kenney selected his other strategy, he must accept the possibility of a less favorable outcome.

Step 5. The Decision

Kenney concentrated his reconnaissance aircraft on the northern route.

Discussion

Let us assume that the Japanese commander used a similar philosophy of decision, basing his decision on his enemy's capabilities. Considering the four battles as sketched, the Japanese commander could select either the left or the right column, but could not select the row. If he sailed the northern route, he exposed the convoy to a maximum of two days of bombing. If he sailed the southern route, the convoy might be subjected to three days of bombing. Since he sought minimum exposure to bombing, he should select the northern route.

These two independent choices were the actual decisions which led to the conflict known in history as the Battle of the Bismarck Sea. Kenney concentrated his reconnaissance on the northern route; the Japanese convoy sailed the northern route; the convoy was sighted approximately one day after it sailed; and Allied bombing started shortly thereafter. Although the Battle of the Bismarck Sea ended in a disastrous defeat for the Japanese, we cannot say the Japanese commander erred in his decision. A similar convoy had reached Lae with minor losses two months earlier. The need was critical, and the Japanese were prepared to pay a high price. They did not know that Kenney had modified a number of his aircraft for low-level bombing and had perfected a deadly technique. The U.S. victory was the result of careful planning, thorough training, resolute execution, and tactical surprise of a new weapon—not of error in the Japanese decision.

These familiar with game theory will recognize that the Rabaul-Lae situation presents all the features of a two-person game. The two commanders have independent choices of action, and these interact to determine a particular battle. The Bismarck-Sea battle exposed the Japanese convoy to a certain number of days of bombing. The 'game' situation
In general, we may not be able to remove any strategy or after the removal of some strategies the game matrix is still quite big. The Principle of Removal of Dominated Strategies can only help us to simplify the game matrix somewhat.
Remark: In the above analysis, we used the basic assumption of Common Knowledge.

A fact is common knowledge if everyone knows it, everyone knows that everyone knows it, everyone knows that everyone knows that everyone knows it,..., and so on ad infinitum.

你知, 我知, 你知我知, 我知你知, ........
• Common knowledge is a phenomenon which underwrites much of social life. In order to communicate or otherwise coordinate their behavior successfully, individuals typically require mutual or common understandings or background knowledge.

• If a married couple are separated in a department store, they stand a good chance of finding one another because their common knowledge of each others' tastes and experiences leads them each to look for the other in a part of the store both know that both would tend to frequent. Since the spouses both love cappuccino, each expects the other to go to the coffee bar, and they find one another.

• In *A Treatise of Human Nature*, Hume argued that a necessary condition for coordinated activity was that agents all know what behavior to expect from one another. Without the requisite mutual knowledge, Hume maintained, mutually beneficial social conventions would disappear.
Once upon a time an evil King decided to grant sadistic amnesty to a large group of prisoners, who were kept incommunicado in the dungeons. The King placed a hat on each prisoner; two of these hats were red, the rest white. The King summoned the prisoners and commanded them not to look upward. Thus each prisoner could see the hat of every one of his fellow prisoners, but not his own. The King spoke thus: “Most of you are wearing white hats, but at least one of you is wearing a red hat. Every day from now on you will be brought to here from your solitary confinement. The day that you guess correctly the color of the hat that you are wearing is the day you will go free. If you guess incorrectly, you will be instantly beheaded.”

**How many days would it take the two red-hatted prisoners to infer, rationally, the color of their hats?**
An honest father tells his two sons that he has placed $10^n$ dollars in one envelope, and $10^{n+1}$ dollars in the other, where $n$ is chosen with equal probability among integers between 1 to 6. The father randomly hands each son an envelope. The first son looks inside and finds $10,000. He calculates that the other envelope contains either $1,000 or $100,000 with equal probability. The expected amount in the other envelope is then $50,500. The second son finds only $1,000 in his envelope. Again, he calculates that the expected amount of the other envelope is $5,050. The father privately asks each son whether he would be willing to pay $1 to switch envelopes. Both son say yes. The father then tells each son what his brother said and repeats the question. Again, both say yes. The father relays the brothers’ answers and ask each a third time. Again both say yes. But if the father relays the answer and ask a fourth time, the son with $1,000 will say yes, but the son with $10,000 will say no.

Why?
Assignment 5:

10. In the following of Simplified Morra, write down the set of strategies for each player and the payoff matrix.

**Simplified Morra:**
Each of two players show one finger or two fingers, and simultaneously guesses how many fingers the other player will show. If both players guess correctly, or both players guess incorrectly, there is no payoff. If just one player guesses correctly, that player wins a payoff equal to the total of fingers shown by both players.
11. Write down the set of strategies for each player and the payoff matrix for the following Colonel Blotto Game.

**Colonel Blotto Games.**

Colonel Blotto has 4 regiments with which to occupy two posts. The famous Lieutenant Kije has 3 regiments with which to occupy the same posts. The payoff is defined as follows. The army sending the most units to either post captures it and all the regiments sent by the other side, scoring one point for the captured post and one for each captured regiment. If the players send the same number of regiments to a post, both forces withdraw and there is no payoff.
We may apply the following two principles to analyze the game.

**Equilibrium Principle**: Best Responses to each other

This principle involves the interactions of the players.

**Maximin Principle**: Safety First

Under this principle, each player only concerns his/her own payoff.
**Equilibrium Principle for pure strategies:**

**Saddle points (PSE):**

If some entry $a_{ij}$ of the matrix $A$ has the property that

1. $a_{ij}$ is the minimum of the $i$th row, and
2. $a_{ij}$ is the maximum of the $j$th column,

then we say $a_{ij}$ is a saddle point. If $a_{ij}$ is a saddle point, then Player I can then win at least $a_{ij}$ by choosing row $i$, and Player II can keep her loss to at most $a_{ij}$ by choosing column $j$.

$<\text{Row } i, \text{Column } j>$ is then a PSE or an equilibrium pair, i.e. BR to each other.
Example:

For the following, the central entry, 2, is a saddle point, since it is a minimum of its row and maximum of its column.

\[
A = \begin{pmatrix}
4 & 1 & -3 \\
3 & 2 & 5 \\
0 & 1 & 6
\end{pmatrix}
\]
For large $m \times n$ matrices it is tedious to check each entry of the matrix to see if it has the saddle point property. We can use the following labeling algorithm to find saddle points.

Labelling Algorithm:

1. Go through the game matrix row by row. Put a star on the entry that is the minimum of its row.

2. Go through the game matrix column by column. Put a star on the entry that is the maximum of its column.

3. The entries with two stars are saddle points.
Example: Labeling algorithm to find saddle points

<Row 2, Column 2> is a saddle point.

\[
A = \begin{pmatrix}
*4 & 1 & -3* \\
3 & *2* & 5 \\
0* & 1 & *6
\end{pmatrix}
\]
Example:

In matrix $\mathbf{A}$, there is no saddle point.

However, if the 2 in position $a_{12}$ were changed to 1, then we have matrix $\mathbf{B}$. Here, the minimum of the fourth row is equal to the maximum of the second column; so $b_{42}$ is a saddle point.

$$\mathbf{A} = \begin{pmatrix}
*3 & *2 & 1 & 0* \\
0* & 1 & *2 & 0* \\
1 & 0* & *2 & 1 \\
*3 & 1* & *2 & *2
\end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix}
*3 & *1 & 1 & 0* \\
0* & *1 & *2 & 0* \\
1 & 0* & *2 & 1 \\
*3 & *1* & *2 & *2
\end{pmatrix}$$
Remark: A game matrix may have several saddle points. However, the values of the saddle points are all equal (Assignment 6). In this case there is a well-defined concept of the value of the game as the value of the saddle point.

For the game of Odd and Even, the game matrix (in below) has no saddle point. Therefore, using the Equilibrium Principle in this context cannot help us to analyze this game.

Instead, we will try the Maximin Principle.

\[
\begin{pmatrix}
-2 & 3 \\
3 & -4
\end{pmatrix}
\]
Maximin Principle means to find the “risk” for each strategy and then find the strategy, called the safety strategy, with minimum risk.

Using this “safety strategy”, one can guarantee to get at least a certain amount of payoff.
If Player I uses Row 1, the worst payoff is -2.

\[ \text{Min} \ A(x_1, y_j) = -2 \]

If Player I uses Row 2, the worst payoff is -4.

\[ \text{Min} \ A(x_2, y_j) = -4 \]

Therefore, the “best” of the worst case is -2.

\[ \text{Max Min} \ A(x_i, y_j) = -2 \]

It is achieved by Row 1. If Player I uses Row 1, he can guarantee to get a payoff of -2.
Adopting the Maximin (Safety) Principle for Player II, we will talk about Minimax.

$$\min_{j} \max_{i} A(x_i, y_j) = 3$$

It can be achieved by using Column 1 or Column 2.
Can we do better?

Suppose Player I flips a fair coin to decide his strategy. If Head appears, he uses Row 1. If Tail appears, he uses Row 2.

Player I is expected to get \((-2)\cdot(0.5)+3\cdot(0.5)=0.5\), if Player II uses Column 1.

Player I is expected to get \((3)\cdot(0.5)+(-4)\cdot(0.5)=-0.5\), if Player II uses Column 2.

Therefore, the worst payoff for this randomized strategy is -0.5.

We should randomize our strategy!
Mixed Strategies

Consider a finite 2-person zero-sum game, $(X, Y, A)$, with $m \times n$ matrix, $A$.

Let us take the strategy space $X$ to be the first $m$ integers,

$$X = \{1, 2, \ldots, m\},$$

and similarly, $Y = \{1, 2, \ldots, n\}$.

A mixed strategy for Player I may be represented by a column vector, $(p_1, p_2, \ldots, p_m)^T$ of probabilities that add to 1.

Similarly, a mixed strategy for Player II is an $n$-tuple

$$q = (q_1, q_2, \ldots, q_n)^T.$$
The sets of mixed strategies of players I and II will be denoted respectively by $X^*$, $Y^*$.

$$X^* = \{ \mathbf{p} = (p_1, \ldots, p_m)^T : p_i \geq 0, \text{ for } i = 1, \ldots, m \text{ and } p_1 + \ldots + p_m = 1 \}$$

$$Y^* = \{ \mathbf{q} = (q_1, \ldots, q_n)^T : q_j \geq 0, \text{ for } j = 1, \ldots, n \text{ and } q_1 + \ldots + q_n = 1 \}$$

$p = (p_1, \ldots, p_m)^T$ means playing Row 1 with probability $p_1$, playing Row 2 with probability $p_2$, ..., playing Row $m$ with probability $p_m$. 
The $m$-dimensional unit vector $e_k \in X^*$ with a one for the $k^{th}$ component and zeros elsewhere may be identified with the pure strategy of choosing row $k$.

**Remark:** $X^*$, $Y^*$ are compact convex sets such that the vertices correspond to pure strategies.
Extension of payoff to mixed strategies:

We may consider the set of Player I’s pure strategies, $X$, to be a subset of $X^*$. Similarly, $Y$ may be considered to be a subset of $Y^*$.

We could if we like consider the game $(X, Y, A)$ in which the players are allowed to use mixed strategies as a new game $(X^*, Y^*, A)$,

where $A(p, q) = p^T A q = p_1 a_{11} q_1 + p_1 a_{12} q_2 + \ldots + p_m a_{mn} q_n$. 
Note that
\[ A(p, q) = p^T Aq = p_1 a_{11} q_1 + p_1 a_{12} q_2 + \ldots + p_m a_{mn} q_n \]
\[ = p_1 A(\text{Row1}, q) + p_2 A(\text{Row2}, q) + \ldots + p_m A(\text{Row m, q}) \]
\[ = q_1 A(p, \text{Col1}) + q_2 A(p, \text{Col 2}) + \ldots + q_n A(p, \text{Col n}). \]

Since \( p_i \) and \( q_j \) are all nonnegative, it is easy to see that the BR to the strategy \( p \) is achieved by a pure strategy (Column), also the BR to the strategy \( q \) is achieved by a pure strategy (Row).
Example:

\[
\begin{array}{ccc}
q & 1-q \\
p & a & b \\
1-p & c & d
\end{array}
\]
Remark:
In this extension, we have made a rather subtle assumption. We assumed that when a player uses a mixed strategy, he is only interested in his average return. He does not care about his maximum possible winnings or losses — only the average.

This is actually a rather drastic assumption. The main justification for this assumption comes from utility theory.

The basic premise of utility theory is that one should evaluate a payoff by its utility to the player rather than on its numerical monetary value. Utility theory is one of the fundamental contributions of von Neumann and Morgenstern.
Remark: There are some philosophical issues about using mixed strategies.

Do we really use mixed strategies in real life?
It is easy to describe the set of mixed strategies when there are two pure strategies, say $x_1$, $x_2$.

Then, the set of mixed strategies is

$$\{(p_1, p_2): 1 \leq p_1 \leq 0, 1 \leq p_2 \leq 0, p_1 + p_2 = 1\}.$$  

We can rewrite the set as $$\{(p, 1-p): 1 \leq p \leq 0\},$$ i.e.
the set of mixed strategies can be identified as the unit interval.

It is easy to find the BR to $(p, 1-p)$ for $1 \leq p \leq 0$ by graphical method.
Suppose Player I has two strategies. Then, a general mixed strategy is of the form \((p, 1-p), 1 \leq p \leq 0\). Note that

\[
A( (p, 1-p), \text{Col 1}) = pa_{11} + (1-p)a_{21}
\]

\[
A( (p, 1-p), \text{Col 2}) = pa_{12} + (1-p)a_{22}
\]

\[
A( (p, 1-p), \text{Col 3}) = pa_{13} + (1-p)a_{23}
\]

\[
A( (p, 1-p), \text{Col n}) = pa_{1n} + (1-p)a_{2n}
\]

are linear functions in \(p\). The graphs are then straight lines.
It is then easy to find the BR columns to \((p, 1-p)\) in the following.

\[
\begin{array}{cccc}
    p & a_{11} & a_{12} & a_{13} \\
1-p & a_{21} & a_{22} & a_{23} \\
    & & & \ldots & a_{1n}
\end{array}
\]

For \(0 \leq p \leq \alpha\), BR column is Col1.
For \(\alpha \leq p \leq 1\), BR column is Col3.
Assignment 6:

11. Write down a 2x2 game matrix with no saddle point.

12. Given the following 2x4 game write down the payoff if Player I is using (1/3, 2/3) and II using (1/6, ¼, ¼, 1/3). What is Player II’s BR to Player I’s (1/3, 2/3)?

\[
\begin{array}{cccc}
3 & 2 & 4 & 0 \\
-2 & 1 & -4 & 5 \\
\end{array}
\]

13. For the 2x4 game in Problem 12, find Player II’s BR to Player I’s strategy \((p, 1-p)\) for \(p\) between 0 and 1.

14. Suppose an \(m \times n\) game has saddle points at \(a_{ij}\) and \(a_{pq}\). Show that \(a_{ij} = a_{pq}\).
Safety strategies:

For each $p \in X^*$, the worst payoff is $\min_{q \in Y^*} p^T A q$.

This minimum is achieved at a pure strategy of Player II. We will find a strategy $p^* \in X^*$ such that $\min_{q \in Y^*} p^T A q$ is the largest. Therefore, we say that $p^*$ achieves the maximin and we denote this value as $\text{MaxMin}$.

This is called the Safety strategy or Maximin strategy for Player I.
For Player II, he/she will find \( q^* \in Y^* \) so that \( \max_{p \in X^*} p^T A q \) is the smallest and we denote this value as \( \text{Min Max} \).

This is called the Safety strategy or the Minimax strategy for Player II.

Remark: We need to use methods in mathematical analysis to guarantee that safety strategies exist. In this case, we need to use the fact that the sets of mixed strategies are compact convex sets.
Question: How to find Safety strategies?
For the case of two strategies, we can use graphical method to find safety strategies.
Question: How to find Safety strategies?
For the case of two strategies, we can use graphical method to find safety strategies.
Example: Odd or Even

\[
\begin{array}{cc}
p & -2 & 3 \\
1-p & 3 & -4 \\
\end{array}
\]

\[\text{Col} 1 \quad p(-2) + (-p)^2 \geq 3\]

\[\text{Col} 2 \quad p(3) + (1-p)(-4) \leq 2\]
Example: Odd or Even

\[
\begin{array}{c|c}
8 & 1-8 \\
\hline
-2 & 3 \\
3 & -4 \\
\end{array}
\]

Row 1: \(3 \cdot (-2) + (1-8) \cdot 3\)

Row 2: \(9 \cdot 3 + (1-8) \cdot 4\)
Example: Battle of Bismark Sea

\[
\begin{pmatrix}
  2 & 2 \\
  1-p & 3 \\
\end{pmatrix}
\]

Col1: \( p_2 + (1-p) \times 1 \)

Col2: \( p_2 + (1-p) \times 3 \)
Note that when there is no saddle point, the safety strategy is achieved at the intersection of two lines. Then it is easy to solve for the safety strategy.

Since the safety strategy is achieved at the intersection of two lines, we can illustrate our result by 2x2 games.

<table>
<thead>
<tr>
<th></th>
<th>q</th>
<th>1-q</th>
</tr>
</thead>
<tbody>
<tr>
<td>p</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>1-p</td>
<td>c</td>
<td>d</td>
</tr>
</tbody>
</table>

Then, to find $p$ we write $|c-d|$ for the first row and $|a-b|$ for the second row. As $p$ must be between 0 and 1, we get $p = \frac{|a-b|}{|a-b| + |c-d|}$

Similarly to find $q$, we write $|b-d|$ for the first column and $|a-c|$ for the second column. As $q$ is between 0 and 1, $q = \frac{|b-d|}{|b-d| + |a-c|}$
Example: Matching Pennies

\[ \begin{array}{c|cc}
   & 1 & -1 \\
---&---&---
   1-p & 1 & -1 \\
   p & -1 & 1 
\end{array} \]

\[ \begin{align*}
   \text{Col1: } & \quad p(1+p)(-1) \\
   \text{Col2: } & \quad p(-1)(1-p)x \\
   \end{align*} \]

Ans: Optimal strategy for Player 1 is \( \left( \frac{1}{2}, \frac{1}{2} \right) \). Value of the game is \( \frac{1}{2} \times 1 + \frac{1}{2} \times 0 = 0 \).
Example: Odd or Even

\[\begin{array}{c|cc}
p & -2 & 3 \\ \hline
1-p & 3 & -4 \\
\end{array}\]

\[p \leq \frac{2}{5} \Rightarrow (1-p) < (-4)\]

\[p \leq \frac{2}{5} \Rightarrow (1-p) < (-4)\]

Ages: Safety strategy for Player I is \((\frac{7}{12}, \frac{5}{12})\)

Value = \(\frac{7}{12} \times (-2) + \frac{5}{12} \times 3 = \frac{1}{12}\)
Example: Odd or Even

\[
\begin{array}{ccc}
9 & 1-9 \\
-2 & 3 \\
3 & -4
\end{array}
\]

Row 2: \(9x3 + (1-9)x4\)

Row 1: \(3x(-2) + (1-9)x3\)

Ans: Safety strategy for Player II is \(\left(\frac{7}{12}, \frac{5}{12}\right)\).

Value of the game: \(\frac{7}{12}(-2) + \frac{5}{12}x3 = \frac{1}{12}\)
Example: Battle of Bismark Sea

\[
\begin{array}{c|cc}
  & 2 & 2 \\
\hline
p & 1 & 3 \\
1-p & & \\
\end{array}
\]

\[
\text{Col1} = \frac{p^2 + (1-p)}{1}
\]

\[
\text{Col2} = \frac{p^2 + (1-p)^3}{3}
\]

This game has a saddle point at (Row 1, Col 1)

Ans: Safety strategy for Player I is Row 1. Safety strategy for Player II is Col 1. Value = 2
Minimax Theorem (John von Neuman, 1927): MinMax=MaxMin

Remark: Because of the Minimax Theorem, the safety strategies are also called the optimal strategies and the value MinMax=MaxMin is called the value of the game.
We can solve two person zero-sum games effectively if either Player I (row chooser) or Player II (column chooser) has only two strategies, i.e. 2xn or mx2 games.

1. Eliminated dominated rows or columns and then look for saddle points. If there is a saddle point then it is the solution of the game.

2. Suppose it is a 2xn game with game matrix

```
| a11  a12  ...  a1n |
| a21  a22  ...  a2n |
```

Plot the graph of the response of the k\textsuperscript{th} column $p_{ak} + (1-p)a_{2k}$.

3. Look for the highest point of the lower envelope of the lines. Suppose the point is the intersection of the response of Column k and Column l. This means that Player II will only use these two columns. We can then solve for optimal (safety) strategies for each player and the value of the game as in 2x2 games.
4. Suppose it is a mx2 game with game matrix
Plot the graph of the response of $i^{th}$ row
$q a_{i1} + (1-q) a_{i2}$.

5. Look for the lowest point of the upper envelope
and let it be the intersection of the response of Row I and
Row j. This means that Player I will only use these two
rows. We can then solve for optimal (safety) strategies and
the value of the game as in 2x2 games.
Example: Solve the following two-strategy games.

\[
\begin{array}{cccc}
2 & 3 & 5 & 1 \\
4 & 1 & 0 & 6 \\
\end{array}
\quad
\begin{array}{cc}
3 & -2 \\
-1 & 4 \\
1 & 2 \\
\end{array}
\]
1. We cannot find any dominated row or column. There is no saddle point for this game matrix.

2. This is a two-strategy game. We will use graphical method to find safety strategies for Player I.
The highest point of the lower envelope occurs at the intersection of Col2 & Col4. So the situation is:

\[
\begin{array}{c|c|c}
5/7 & 3/7 & 2/7 \\
\hline
5 & 3 & 1
\end{array}
\]

Safety strategy for Player I: \((\frac{5}{7}, \frac{3}{7})\)
Safety strategy for Player II: \((0, \frac{5}{7}, 0, \frac{3}{7})\)

Value = \(\frac{5}{7} \times 3 + \frac{3}{7} \times 1 = \frac{17}{7}\)
1. We cannot find any dominated row or column and there is no saddle point.
2. This is a two-strategy game. We will use graphical method to find strategy for Player II.

The lowest point of the upper envelope occurs at the intersection of Row 1 & Row 3. So the situation is:

\[
\begin{array}{ccc}
\frac{1}{6} & -2 & \frac{5}{6} \\
\frac{5}{6} & 1 & 2 \\
\end{array}
\]

Ans: Safety strategy for Player I: \((\frac{1}{6}, 0, \frac{5}{6})\)

Safety strategy for Player II: \((\frac{4}{6}, \frac{3}{6})\)

Value = \(3 \times \frac{5}{6} + (2) \frac{2}{6} = \frac{4}{3}\)
Assignment 7:

15. Solve the following 2-strategy games. Write down the value of the game and the optimal (safety) strategy for each player.

(a). \[
\begin{pmatrix}
-1 & -3 \\
-2 & 2
\end{pmatrix}
\]

(b). \[
\begin{pmatrix}
0 & 10 \\
1 & 2
\end{pmatrix}
\]

(c). \[
\begin{pmatrix}
3 & -1 & 1 \\
-2 & 4 & 2
\end{pmatrix}
\]

(d). \[
\begin{pmatrix}
1 & 5 \\
5 & 4 \\
6 & 3
\end{pmatrix}
\]