Homework 3

§4.1: 2, 8, 16, 18, 20
§4.2: 6, 8, 24, 32
§4.3: 10, 14, 20, 24
§4.4: 4, 8, 14, 30
§4.5: 12, 14, 30
§4.6: 4, 6, 14, 28

Supplementary exercises for determinants

1. Let \( \sigma = 364152, \tau = 246513, \rho = 413562 \) be permutations of \( \{1, 2, 3, 4, 5, 6\} \).
   (a) Find parity of \( \sigma, \tau, \rho \).
   (b) Find \( \tau \circ \sigma, \rho \circ \tau \circ \sigma \), and \( \sigma^{-1} \).

2. Let \( g = g(x_1, \ldots, x_n) = \prod_{i<j}(x_i - x_j) \). Let
   \[
   \sigma(g) = \prod_{i<j}(x_{\sigma(i)} - x_{\sigma(j)}).
   \]
   Show that \( \sigma(g) = (\text{sgn} \sigma)g \).

3. Find the determinant of each of the following matrices.
   \[
   A = \begin{bmatrix}
   7 & 6 & 5 \\
   2 & 1 & 1 \\
   3 & 2 & 1
   \end{bmatrix}, \quad
   B = \begin{bmatrix}
   -2 & -1 & 4 \\
   6 & -3 & -2 \\
   4 & 1 & 2
   \end{bmatrix}, \quad
   C = \begin{bmatrix}
   2 & 1 & 3 & 2 \\
   3 & 0 & 1 & -2 \\
   1 & -1 & 4 & 3 \\
   2 & -2 & -1 & 1
   \end{bmatrix}
   \]

   Then find the adjoint \( \text{adj} A \), \( \text{adj} B \), \( A^{-1} \), and \( B^{-1} \) if \( A \) and \( B \) are invertible.

4. Let \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) be defined by \( T(x_1, x_2, x_3) = (x_1 + x_2 - x_3, x_1 - x_2 + x_3, -x_1 + x_2 + x_3) \). Let \( P \) be the parallelepiped spanned by the three vectors
   \[
   \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad
   \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad
   \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},
   \]
   i.e., \( P = \{c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 \mid 0 \leq c_1, c_2, c_3 \leq 1\} \). Find the volume of \( P \) and the volume of \( T(P) = \{T(x) \mid x \in P\} \).

Supplementary exercises for matrices of linear transformations

1. Let \( \mathbb{P}_n(t) \) be the vector space of all polynomials of degree at most \( n \). Let
   \[
   \mathcal{B} = \{1, t, t(t+1), t(t+1)(t+2)\}, \quad \mathcal{B}' = \{1, t, t(t-1), t(t-1)(t-2)\}.
   \]
   (a) Show that \( \mathcal{B} \) and \( \mathcal{B}' \) are bases of \( \mathbb{P}_3(t) \).
   (b) Find the transition matrix from \( \mathcal{B} \) to \( \mathcal{B}' \).
   (c) Show that \( T : \mathbb{P}_3(t) \to \mathbb{P}_3(t) \), defined by \( T(p(t)) = p(t) + tp'(t) \), is a linear transformation.
   (d) Find the matrix \( A \) of \( T \) relative to the basis \( \mathcal{B} \), and the matrix \( B \) of \( T \) relative to the basis \( \mathcal{B}' \).
   (e) Find the relation between the matrices \( A \) and \( B \).

2. Let \( \mathcal{M}_{m,n} \) be the vector space of all \( m \times n \) matrices. Note that
   \[
   \mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}
   \]
   is a basis of \( \mathcal{M}_{3,2} \), and
   \[
   \mathcal{C} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}
   \]
   is a basis of \( \mathcal{M}_{2,2} \).
(a) Show that the set
\[ B' = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \]
is a basis of \( M_{3,2} \), and that
\[ C' = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\} \]
is a basis of \( M_{2,2} \).

(b) Show that \( F : M_{3,2} \rightarrow M_{2,2} \), defined by
\[ F \left( \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix} \right) = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix}, \]
is a linear transformation.

(c) Find the matrix \( A \) of \( F \) relative to the bases \( B \) and \( C \), and the matrix \( B \) of \( F \) relative to the bases \( B' \) and \( C' \).

(d) Find the relation between the matrices \( A \) and \( B \).

3. Let \( F : M_{2,2} \rightarrow \mathbb{P}_2(t) \) be defined by
\[ F \left( \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \right) = (x_{11} + x_{12}) + (x_{12} + x_{21})t + (x_{21} + x_{22})t^2. \]

(a) Find the matrix \( A \) of \( F \) relative to the basis
\[ B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \]
of \( M_{2,2} \) and the basis \( C = \{1, t, t^2\} \) of \( \mathbb{P}_2 \).

(b) Find the matrix \( B \) of \( F \) relative to the basis
\[ B' = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\} \]
of \( M_{2,2} \) and the basis \( C' = \{1, t, t(t+1)\} \) of \( \mathbb{P}_2 \).

(c) Find the relation between \( A \) and \( B \).

4. Let \( T : \mathbb{R}^5 \rightarrow \mathbb{R}^3 \) be a linear transformation defined by
\[ T(x_1, x_2, x_3, x_4, x_5) = (x_1 + x_2 + x_4, x_2 - x_3 + x_5, 3x_1 + 3x_3 - 2x_4 + 2x_5). \]

(a) Find the matrix \( B \) of \( T \) relative to the basis
\[ B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\} \]
of \( \mathbb{R}^5 \) and the basis
\[ C = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\} \]
of \( \mathbb{R}^3 \).
(b) Let $V$ be the subspace of $\mathbb{R}^5$ defined by the linear system

\[
\begin{align*}
&x_1 + x_2 + x_3 + x_4 + x_5 = 0 \\
&x_1 + x_2 - x_3 - x_4 + x_5 = 0
\end{align*}
\]

and let $W$ be the subspace of $\mathbb{R}^3$ defined by the linear equation $2x_1 + 3x_2 + x_3 = 0$. Show that $T$ defines a linear transformation from $V$ to $W$.

(c) Find the matrix of $T$ from $V$ to $W$ relative to the basis $B$ of $V$, consisting of the basic solutions of the linear system (*), and the basis $C$ of $W$, consisting of the basic solutions of the linear equation $2x_1 + 3x_2 + x_3 = 0$.

5. Let $V$ be an $n$-dimensional vector space, and let $W$ be an $m$-dimensional vector space. Let $\text{Hom}(V, W)$ denote the set of all linear transformations from $V$ to $W$. For $F, G \in \text{Hom}(V, W)$, define the addition and scalar multiplication as

\[
(F + G)(v) = F(v) + G(v),
\]

\[
(cF)(v) = cF(v).
\]

(a) Show that $\text{Hom}(V, W)$ is a vector space.

(b) Given a basis $B$ of $V$ and a basis $C$ of $W$. Let $T : \text{Hom}(V, W) \longrightarrow M_{m,n}$ be defined by

\[
T(F) = \text{the matrix of } F \text{ relative to the bases } B \text{ and } C.
\]

Show that $T$ is a one-to-one and onto linear transformation.

6. Let $V$ be the set of functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ of the form $f(t) = (a_0 + a_1t + a_2t^2)e^{2t}$, where $a_0, a_1, a_2 \in \mathbb{R}$.

(a) Show that $V$ is a subspace of the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$.

(b) Let $D : V \longrightarrow V$ be defined by $D(f(t)) = f'(t)$. Find the matrix of $D$ relative to the basis \{\(e^{2t}, te^{2t}, t^2e^{2t}\}\).

(c) Is $D$ invertible? If yes, find the inverse transformation of $D$. 

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