1. (15 pts)
   (a) Solve the system of linear equations:

   \[
   \begin{align*}
   x_3 + 2x_4 + 3x_5 &= 0 \\
   x_1 + 2x_2 + 2x_3 + 2x_4 + 3x_5 &= 0 \\
   2x_1 + 4x_2 + x_3 - 2x_4 - 3x_5 &= 0 \\
   2x_1 + 4x_2 + 3x_3 + 2x_4 + 3x_5 &= 0
   \end{align*}
   \]

   \( (1) \)

   (b) Find a basis for the orthogonal complement of the solution space of the linear system (1) in \( \mathbb{R}^5 \).

   (c) Is there any linear transformation \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^5 \) such that the range of \( T \) is the solution space of the linear system (1)? If yes, find the standard matrix of such a linear transformation \( T \).
2. (15 pts) Let \( \mathbf{P}_3 \) be the vector space of all polynomials in one variable \( t \) of degree at most 3. Note that \( \mathcal{B} = \{1, t, t^2, t^3\} \) is a basis of \( \mathbf{P}_3 \). Let

\[
\begin{align*}
  f_1(t) &= 1 + t + 2t^2 + 3t^3, \\
  f_2(t) &= 2 + 2t + 4t^2 + 6t^3, \\
  f_3(t) &= 1 + 4t^2 + 6t^3, \\
  f_4(t) &= 3 + 5t + 2t^2 + 3t^3, \\
  f_5(t) &= 2 + 3t + 3t^2 + 3t^3.
\end{align*}
\]

(a) Find the coordinate vectors for \( f_1(t), f_2(t), f_3(t), f_4(t), f_5(t) \) relative to the basis \( \mathcal{B} \) respectively.

(b) Find a basis for \( \text{Span}\{f_1, f_2, f_3, f_4, f_5\} \) from the set \( \{f_1, f_2, f_3, f_4, f_5\} \) by row operations.

(c) Determine the dimension of \( \text{Span}\{f_1, f_2, f_3, f_4, f_5\} \).
3. (15 pts) Let $A = \begin{bmatrix} 3a & 4 & 2 \\ -6 & 2a & 3 \\ 12 & 2 & 1 \end{bmatrix}$, where $a$ is an unspecified parameter.

(a) Calculate $\text{det } A$.

(b) Find all values for $a$ so that the matrix $A$ has zero as an eigenvalue.

(c) Find all values for $a$ so that the rank of $A$ is 3.
4. (10 pts) Let $B = \{v_1, v_2, v_3\}$ and $C = \{w_1, w_2, w_3\}$ be two bases for a subspace $W$ of $\mathbb{R}^4$, where

$$
v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix}; \quad w_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ -2 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \quad w_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}.
$$

(a) Let $v$ be a vector in $W$ whose coordinate vector relative to the basis $C$ is $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$. Find the coordinate vector of $v$ relative to the basis $B$.

(b) (Not for students in Prof. Qian’s Lecture.) If a vector $w$ in $W$ has the coordinate vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ relative to the basis $C$, find its coordinate vector relative to the basis $B$ in terms of $a, b, c$. 


5. (15 pts) Let \( A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 5 \\ 0 & 0 & 2 \end{bmatrix} \).

(a) Is \( A \) diagonalizable? If yes, find a matrix \( P \) such that \( P^{-1}AP \) is a diagonal matrix.

(b) Compute the matrix \( A^{100} \).

(c) Let \( Q \) be an invertible \( 3 \times 3 \) matrix. Is the matrix \( B = QAQ^{-1} \) diagonalizable? If yes, find a matrix \( U \) such that \( U^{-1}BU \) is a diagonal matrix.
6. (5 pts) Let $A$ be a $3 \times 3$ matrix having distinct eigenvalues $1, -1, 0$. Show that $A^3 = A$. 
7. (15 pts)

(a) Let $W$ be a subspace of $\mathbb{R}^4$ defined by the linear system $Ax = 0$, where

$$A = \begin{bmatrix}
1 & -1 & -1 & 1 \\
1 & -1 & -1 & 1 \\
1 & -1 & -1 & 1 \\
1 & -1 & -1 & 1
\end{bmatrix}.$$

Find the shortest distance from the point $(1, 2, 2, -5)$ to $W$.

(b) Find the standard matrix of the orthogonal projection $\text{Proj}_W : \mathbb{R}^4 \to \mathbb{R}^4$. 

8. (10 pts) Let $W$ be a subspace of $\mathbb{R}^4$ with a basis $\mathcal{B} = \{a_1, a_2, a_3\}$, where
\[
a_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 0 \\ 6 \\ 3 \\ -3 \end{bmatrix}.
\]

(a) Find an orthogonal basis for $W$ by the Gram-Schmidt process.

(b) Find a $QR$-decomposition of the matrix $A$ whose column vectors are $a_1, a_2, a_3$. 
