Problem Set 5 – Graph Theory

1. Let G be a simple graph with n vertices and m edges. Show that $2m \leq n(n-1)$.

2. Show that the number vertices of odd degree of any graph is an even number.

3. A vertex in a digraph $D = (V, A)$ is called a **source** (**sink**) if its in-degree (out-degree) is zero. A **directed cycle** of D is a cycle whose in-degree and out-degree at every vertex are exactly one. Show that any nontrivial digraph without sources and sinks has at least one directed cycle.

4. Let G be a simple graph with n vertices. Its **complement** is the simple graph with the same vertex set and two vertices are adjacent in $\bar{G}$ if and only if they are not adjacent in G. Such a graph is called **self-complementary**. Find examples of a self-complementary graph on four vertices and on five vertices.

5. A saturated hydrocarbon is represented by a structural formula in which each C atom has degree 4 and each H has degree 1. Show that the hydrocarbon is acyclic (has no carbon rings in it) if and only if its structural formula is of the form C_nH_{2n+2}.

6. For each of the following problems, determine whether the relation R on the set A is a tree. If it is a tree, find its leaves.

   (a) $A = \{a, b, c, d, e, f\}$, $R = \{(a, b), (c, e), (f, a), (f, c), (f, d)\}$.

   (b) $A = \{u, v, w, x, y, z\}$, $R = \{(u, x), (u, v), (w, v), (x, z), (x, y)\}$.

7. Performing the preorder search and postorder search to the tree the rooted tree (T, v_0) below.

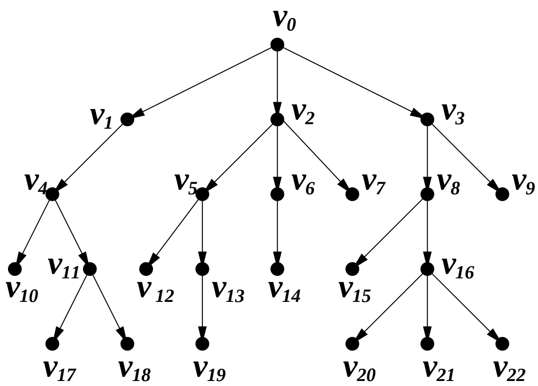

Figure 1: A labeled rooted tree

8. Show that the maximum number of vertices in a binary tree of height n is $2^{n+1} - 1$.

9. Let T be a complete m-ary tree.

   (a) If T has exactly three levels. Prove that the number of vertices of T must be $1 + km$, where $2 \leq k \leq m + 1$.

   (b) If T has n vertices of which k are non-leaves and l are leaves. Prove that $n = mk + 1$ and $l = (m-1)k + 1$.

10. Use Polish notations to construct the trees for the following expressions.

    (a) $(((2 \times 7) + x) \div y) \div (3 - 11)$

    (b) $(3 - (2 - (11 - (9 - 4)))) \div (2 + 3(+4(+7)))$

11. Draw binary trees, respectively, whose preorder, postorder, and inorder searches produce the string: ABCDEFGH.

12. Find a Minimum Spanning Tree for the connected graph below by the Kruskal Algorithm and the Prim Algorithm respectively.

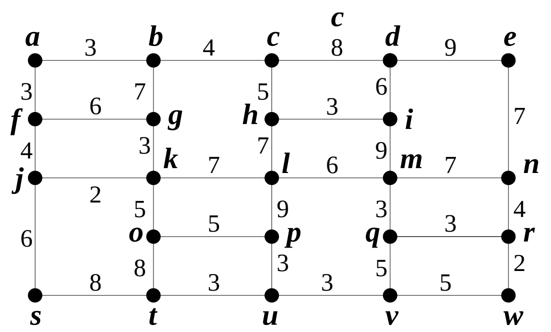

Figure 2: A connected graph

13. Modify Kruskal's and Prim's algorithms so that they will produce a Maximum Spanning Tree, that is, one with the largest possible sum of the weights.

14. Apply Depth-First Search and Breadth-First Search to find a rooted tree for the graph in Figure 2.

15. Find an Euler tour or Euler path for the following graph.

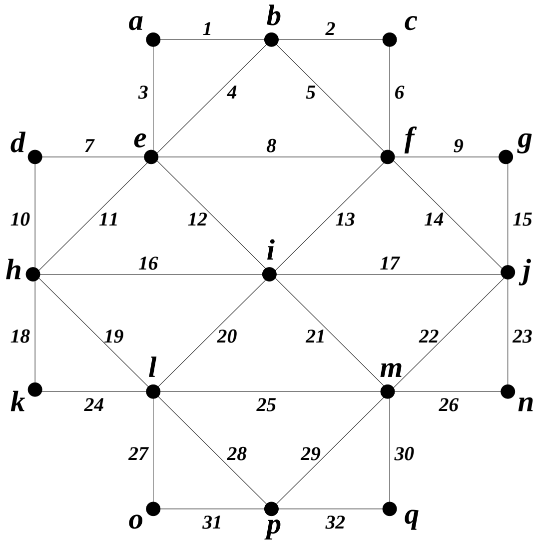

Figure 3: A connected graph

16. Let G be a graph whose vertex set $V(G) = \{1, 2, \ldots, 15\}$ and two vertices $i + j$ is a multiple of 3. Let R be an equivalence relation on $V(G)$ defined by iRj if and only if $i \equiv j \pmod 7$. Find the quotient graph G/R.

17. Let $G = (V, E)$ be a graph. Define a relation R on V by uRv if $u = v$ or if there is a path in G from u to v. Show that R is an equivalence relation.

18. Let $K_{m,n}$ denote the complete bipartite graph with $m, n \geq 2$. (1) How many distinct cycles of length 4 are there in $K_{m,n}$? (2) How many different paths of length 2 are there in $K_{m,n}$? (3) How many different paths of length 3 are there in $K_{m,n}$?

19. Let Q_n be the graph obtained from the n-dimensional unit cube $[0,1]^n$, whose vertices and edges are the vertices and edges of the n-cube $[0,1]^n$. The graph Q_n can be also defined as follows: $V(Q_n)$ is the set of zero-one sequences of length n, and two such sequences are adjacent if and only if they differ at only one position. (1) For which n the graph Q_n has an Euler tour or an Euler path? (2) For what n the graph Q_n is non-planar? Why? (3) For what n the graph Q_n has a Hamilton cycle or Hamilton path? (4) Is Q_n bipartite?

20. Show that the Peterson graph is not planar.

21. A buckyball (or football) is a graph whose every vertex has degree 3 and every face is either a pentagon or a hexagon. Find the numbers of vertices, edges, and faces (pentagons and hexagons) respectively.