1. (a) Verify the following

\[ 25 \equiv 67 \pmod{21}, \quad 3 \cdot 25 \equiv 3 \cdot 67 \pmod{21}, \quad 3 \cdot 14 \equiv 3 \cdot 28 \pmod{21}, \quad 14 \not\equiv 28 \pmod{21}. \]

(b) Let \( a \) be an integer. Verify that for any integers \( k, l \in \mathbb{Z} \),

\[ a + kn \equiv a + ln \pmod{n}. \]

(c) Find all integer solutions for the equation \( x \equiv a \pmod{n} \). In particular, find solutions for \( x \equiv 0 \pmod{12} \) and for \( x \equiv 5 \pmod{12} \).

2. Find all integer solutions for each of the following equations.

(a) \( x \equiv 0 \pmod{6} \); (b) \( 2x \equiv 0 \pmod{6} \); (c) \( 3x \equiv 0 \pmod{6} \); (d) \( 4x \equiv 0 \pmod{6} \);
(e) \( 5x \equiv 0 \pmod{6} \); (f) \( 6x \equiv 0 \pmod{6} \); (g) \( 7x \equiv 0 \pmod{6} \); (h) \( −x \equiv 0 \pmod{6} \).

Then indicate equations having the same solutions.

3. List all invertible integers modulo 15 for the numbers 0, 1, 2, \ldots, 14, and their inverses. Point out integers that are not invertible modulo 15.

4. Solve each of the following equations.

(a) \( 6x \equiv 15 \pmod{12} \); (b) \( 3x \equiv 15 \pmod{12} \); (c) \( 7x \equiv 15 \pmod{12} \); (d) \( 5x \equiv 15 \pmod{12} \).

5. Solve each of the following equations.

(a) \( \begin{cases} x \equiv 0 \pmod{5} \\ x \equiv 0 \pmod{7} \end{cases} \)

(b) \( \begin{cases} x \equiv 0 \pmod{15} \\ x \equiv 0 \pmod{21} \end{cases} \)

(c) \( \begin{cases} x \equiv 18 \pmod{5} \\ x \equiv 30 \pmod{7} \end{cases} \)

(d) \( \begin{cases} x \equiv 18 \pmod{15} \\ x \equiv 30 \pmod{21} \end{cases} \)

6. Consider the statement “If \( 1 = 4 \), then \( 1 = 2 \).” and its following proof

“Since \( 1 + 3 = 4 \) and \( 1 = 4 \), we have \( 0 = 3 \). Dividing both sides of \( 0 = 3 \) by 3, we further have \( 0 = 1 \). Hence \( 1 = 0 + 1 = 1 + 1 = 2 \).”

Is there anything wrong with the proof? That is, whether the proof is a true logic argument? What can you conclude from the statement and proof?

7. Define connectives “\( \downarrow \)” and “\( \Delta \)” in the following table:

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( p \downarrow q )</th>
<th>( p \Delta q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
</tbody>
</table>

Find the truth tables for the statements

\( (p \downarrow q) \downarrow r, \quad (p \downarrow q) \land (p \downarrow r), \quad (p \downarrow q) \downarrow (p \downarrow r), \quad (p \land q) \Delta p, \quad (p \Delta q) \Delta (q \Delta r). \)

8. Let

\[ p : \text{John is a student of Computer Science Department in HKUST.} \]
\[ q : \text{John takes course Math132} \]

(a) Write the English sentences of the converse, inverse, and contrapositive forms for the statement \( p \rightarrow q \).

(b) Write the English sentences for the statements \( \neg p \lor q \) and \( \neg q \lor p \) respectively.

9. Express \( p \downarrow q \) and \( p \Delta q \) in terms of \( p, q \), and connectives \( \neg \) and \( \land \).

10. Express each of the following statements by a Boolean function of Boolean variables \( p, q, r \).

\( (p \downarrow q) \downarrow r, \quad (p \downarrow q) \land (p \downarrow r), \quad (p \downarrow q) \downarrow (p \downarrow r), \quad (p \land q) \Delta p, \quad (p \Delta q) \Delta (q \Delta r). \)
11. Show that the statement
\[(\forall x P(x)) \lor (\forall x Q(x)) \rightarrow \forall x (P(x) \lor Q(x))\]
is a tautology. Show that its converse is not a tautology by giving a counterexample.

12. Show that if statements \(p\) and \(p \rightarrow q\) are tautologies then \(q\) is a tautology. Give an example in daily life for the argument.

13. If \(p \rightarrow q\) and \(q \rightarrow r\) are tautologies, then \(p \rightarrow r\) is a tautology.

14. If \(p \rightarrow q\) and \(\neg q\) are tautologies, then \(\neg p\) is a tautology.

15. Determine whether the conclusion follows logically from the premises, and explain why by using propositional calculus.

   (a) Premises: \[
   \begin{align*}
   &\text{All soldiers can march.} \\
   &\text{Babies are not soldiers.}
   \end{align*}
   \]
   Conclusion: Babies cannot march.

   (b) Premises: \[
   \begin{align*}
   &\text{(1) If Mary goes to Mainland then Nancy is not appointed to Chairman or} \\
   &\text{Oliver is appointed to Secretary.} \\
   &\text{(2) Nancy is appointed to Chairman if Oliver is not appointed to Secretary.} \\
   &\text{(3) Mary goes to Mainland and Nancy is not appointed to Chairman.}
   \end{align*}
   \]
   Conclusion: Oliver is appointed to Secretary.
1. (a), (b) Trivial. (c) $x = a + kn$, $k \in \mathbb{Z}$; $x = 12k$, $k \in \mathbb{Z}$; $x = 5 + 12k$, $k \in \mathbb{Z}$.
2. (a) $x = 6k$, $k \in \mathbb{Z}$; (b) $x = 3k$, $k \in \mathbb{Z}$; (c) $x = 2k$, $k \in \mathbb{Z}$; (d) $x = 3k$, $k \in \mathbb{Z}$; (e) $x = 6k$, $k \in \mathbb{Z}$;
(f) $x = k$, $k \in \mathbb{Z}$; (g) $x = 6k$, $k \in \mathbb{Z}$; (h) $x = 6k$, $k \in \mathbb{Z}$.
3. Invertible elements: 1, 2, 4, 7, 8, 11, 13, 14. Non-invertible elements: 0, 3, 5, 6, 9, 10, 12. Moreover, 1
4. (a) No solution; (b) $x = 1 + 4k$, $k \in \mathbb{Z}$; (c) $x = 9 + 12k$, $k \in \mathbb{Z}$; (d) $x = 3 + 12k$, $k \in \mathbb{Z}$.
5. (a) $x = 35k$, $k \in \mathbb{Z}$; (b) $x = 105k$, $k \in \mathbb{Z}$; (c) $x = 23 + 35k$, $k \in \mathbb{Z}$; (d) $x = 93 + 105k$, $k \in \mathbb{Z}$;
(e) $x = 12 + 35k$, $k \in \mathbb{Z}$; (f) $x = 12 + 105k$, $k \in \mathbb{Z}$.
6. The proof, I mean the argument, is mathematically correct. However, the condition “1 = 4” is an
7. Trivial.
8. Trivial.
9. $p \downarrow q \iff \neg p \land \neg q$; $p \Delta q \iff \neg (\neg p \land \neg q) \land (p \land q)$.
10. $(p \downarrow q) \downarrow r \iff (p \lor q) \land \neg r$; $(p \downarrow q) \land (p \downarrow r) \iff \neg p \land \neg q \land \neg r$;
$(p \downarrow q) \downarrow (p \downarrow r) \iff p \lor (q \land r)$;
$(p \land q) \Delta p \iff p \land q$; $(p \Delta q) \Delta (q \Delta r) \iff (p \land q \land r) \lor (p \land q \land \neg r) \lor (\neg p \land q \land r)$.
11. See lecture notes. Let $I = \{1, 2\}$ and write $p_i = P(i)$, $q_i = Q(i)$ for $i \in I$. Then
   $(\forall i \in I, P(i)) = p_1 \land p_2$; $(\forall i \in I, Q(i)) = q_1 \land q_2$. $(\forall i \in I, P(i) \lor Q(i)) = (p_1 \lor q_1) \land (p_2 \lor q_2)$.
   However, $(p_1 \lor q_1) \land (p_2 \lor q_2) \rightarrow (p_1 \land p_2) \lor (q_1 \land q_2)$ has the false value $F$ when
   $p_1 = q_2 = T$ and $p_2 = q_1 = F$. Thus $(p_1 \lor q_1) \land (p_2 \lor q_2) \rightarrow (p_1 \land p_2) \lor (q_1 \land q_2)$ can not be a tautology.
12. Since $p = T$ and $(p \rightarrow q) = T$, then by definition of $p \rightarrow q$, the statement $q$ must have the true value $T$.
   So $q$ is a tautology.
13. Sine $(p \rightarrow r) = T$ and $(q \rightarrow r) = T$, we have two cases: either $p = T$ or $p = F$. In the formal case, it
   follows $r = T$. Thus by definition, $(p \rightarrow r) = T$. In the latter case, by definition again, $(p \rightarrow r) = T$.
   So $(p \rightarrow r)$ is a tautology.
14. Since $(p \rightarrow q) = T$ and $\neg q = T$, then $(p \rightarrow q) = T$ and $q = F$. By definition, $p = F$. Thus $\neg p = T$.
   So $\neg p$ is a tautology.
15. (a) The conclusion is correct but it does not follow from the premises.
   (b) Let
   $p :$ Mary goes to Mainland.
   $q :$ Nancy is appointed to Chairman.
   $r :$ Oliver is appointed to Secretary.
   It needs to verify whether
   $p \rightarrow \neg q \lor r$
   $\neg r \rightarrow q$
   $p \land \neg q$
   $r$
is a valid inference. Suppose $r = F$, then $\neg r = T$. Thus it follows from $\neg r \rightarrow q$ that $q = T$. However,
   $\neg q = T$. This is a contradiction. So $r = T$. This means that $r$ is indeed a logic conclusion from the premises.