Math132: Problem Set 4

(Deadline: Friday, 29 Oct. 2004)

1. A computer user name consists of three English letters followed by five digits. How many different user names can be made?

2. A set lunch includes a soup, a main course, and a drink. Suppose a customer can select from three kinds of soup, five main courses, and four kinds of drink. How many varieties of set lunches can be possibly made?

3. Find a procedure to determine the number of zeros at the end of the integer $n!$ written in base 10. Justify your procedure and make examples for 12! and 26!.

4. How many different words can be made by rearranging the order of letters in $HONGKONG$?

5. A bookshelf is to be used to exhibit ten mathematics books. There are eight kinds of books on *Calculus*, six kinds of books on *Linear Algebra*, and five kinds of books on *Discrete Mathematics*. Books of the same subject should be displayed together.

   (a) In how many ways can ten distinct books be exhibited so that there are five *Calculus* books, three *Linear Algebra* books, and two *Discrete Mathematics* books?

   (b) In how many ways can ten books (not necessarily distinct) be exhibited so that there are five *Calculus* books, three *Linear Algebra* books, and two *Discrete Mathematics* books?

6. There are n men and n women to form a circle (line), $n \geq 2$. Assume that all men are indistinguishable, all women are also indistinguishable, but each man is distinguishable from each woman.

   (a) How many possible patterns of circles (lines) could be formed so that men and women alternate?

   (b) How many possible patterns of circles (lines) could be formed so that each man is next to at least one woman?

7. Four identical six-sided dice are tossed simultaneously and numbers showing on the top faces are recorded as a multiset of four elements. How many different multisets are possible?

8. Find the number of non-decreasing coordinate paths from the origin $(0, 0, 0)$ to the lattice point (a, b, c).

9. How many six-card hands can be dealt from a deck of 52 cards?

10. How many different eight-card hands with five red cards and three black cards can be dealt from a deck of 52 cards?

11. Fortune draws are arranged to select six ping pang balls simultaneously from a box in which 20 are orange and 30 are white. A draw is lucky if it consists of three orange and thee white balls. What is the chance of a lucky draw?

12. Determine the number of integer solutions for the equation

$$x_1 + x_2 + x_3 + x_4 \leq 38,$$

   where

   (a) $x_i \geq 0$, $1 \leq i \leq 5$.

   (b) $x_1 \geq 0$, $x_2 \geq 2$, $x_3 \geq -2$, $3 \leq x_4 \leq 8$.

13. Determine the number of nonnegative integer solutions to the pair of equations

$$x_1 + x_2 + x_3 = 8, \quad x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 18.$$

14. Show that there must be 90 ways to choose six numbers from 1 to 15 so that all the choices have the same sum.

15. Show that if five points are selected in a square whose sides have length 2, then there are at leat two points whose distance is at most $\sqrt{2}$.

16. Prove that if any 14 numbers from 1 to 25 are chosen, then one of them is a multiple of another.

17. Twenty disks labelled 1 through 20 are placed face down on a table. Disks are selected (by a player) one at a time and turned over until 10 disks have been chosen. If the labels of two disks add up to 21, the player loses. Is it possible to win this game?

18. Show that it is impossible to arrange the numbers $1, 2, \ldots, 10$ in a circle so that every triple of consecutively placed numbers has a sum less than 15.

19. Find the number of ways to arrange the letters E, I, M, O, T, U, Y so that YOU, ME and IT would not occur.

20. Six passengers have a trip by taking a van of six seats. Passengers randomly select their seats. When the van stops for a break, every passenger will leave the van.

    (a) What is the chance that the seat of every passenger after a break is the same as their seat before the break?

    (b) What is the chance that exactly five passengers have the same seats before and after a break?

    (c) What is the probability that at least one passenger has the same seat before and after a break?

21. (Not required) Find the number of nondecreasing lattice paths from the origin $(0, 0)$ to a non-negative lattice point (a, b), allowing only horizontal, vertical, and diagonal unit moves; that is, allowing moves

$$\begin{aligned}(x, y) &\to (x + 1, y),\\ (x, y) &\to (x, y + 1),\\ (x, y) &\to (x + 1, y + 1).\end{aligned}$$

    Hint: For any such path with k diagonal moves ($0 \le k \le \min\{a, b\}$), the number of horizontal moves should be $a - k$ and the number of vertical moves should be $b - k$. Thus

$$\text{answer: } \sum_{k=0}^{\min\{a,b\}} \binom{a+b-k}{a-k, b-k, k}.$$

22. (Not required) **Thinking problem.** Find the number of nondecreasing lattice paths from the origin $(0, 0)$ to a nonnegative lattice point (a, b), allowing arbitrary straight moves from one lattice point to another lattice point so that no lattice points on the line between two lattice points; that is, allowing all moves

$$(x, y) \to (x + k, y + h),$$

    where $k, h \in \mathbb{N}$, $(h, k) \neq (0, 0)$, $\gcd(k, h) = 1$.
    (Answer: unknown)