Math132 Week3: Propositional Logic

Beifang Chen

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1 Statements

By a mathematical statement or just statement we mean a declarative sentence that is either true or false, but not both. The truth value (true or false) for any statement can be determined and is not ambiguous in any sense. For example, the following sentences are statements.

Today is 1st July 1997.

The course number of Discrete Structure in HKUST is Math132.

The equation $x^2 + y^2 = z^2$ has no positive integer solutions.

There are 7,523,804 people in Hong Kong.

However, many sentences in daily life languages are not mathematical statements. For instance, the following sentences are not statements.

How are you?

Hong Kong is a big city.

What a beautiful campus!

This sentence is false.

For the last sentence above, if we say that the sentence is true, then it is false. If, on the other hand, we claim that the sentence is false, then it is true. Such sentences will not be considered as mathematical statements. Statements are usually denoted by small lowercase letters such as $p, q, r, \ldots$, etc.

2 Connectives

Given several statements, we wish to set up rules by which we can decide the truth of various combinations of the given statements. New statements can be formed by using connectives “not”, “and”, and “or”.

The Negation of a statement $p$ is the statement “not $p$”, denoted $\neg p$. The truth values of $\neg p$ are given by the table

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<th>$p$</th>
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The **conjunction** of two statements \( p \) and \( q \) is the statement “\( p \) and \( q \)”, denoted \( p \land q \). Its truth values are given by the table

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The **disjunction** of statements \( p \) and \( q \) is the statement “\( p \) or \( q \)”, denoted \( p \lor q \). Its truth table is

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The **conditional implication** from a statement \( p \) to a statement \( q \) is the statement “if \( p \), then \( q \).” The statement \( p \) is called the **hypothesis** of this implication and \( q \) the **conclusion**. This logical connector is symbolized by \( p \rightarrow q \), and its truth table is defined by

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Whenever \( p \) is false, the implication is irrelevant and the argument is valid for any conclusion, thus it was assigned the true value.

**Example** Let

\[
\begin{align*}
p : & \quad \text{It is a week day.} \\
q : & \quad \text{I go to school.}
\end{align*}
\]

Then the statement \( p \rightarrow q \) is the sentence

If it is a week day, then I go to school.

If it is really a week day, I must go to school; the statement \( p \rightarrow q \) receives a true value. Suppose I did not go to school when it is a week day, there is something wrong; the statement \( p \rightarrow q \) receives a false value. However, if it is not a week day (say weekend or holiday), I don’t need go to school, so it is all right either I go to school or not go to school; the statement \( p \rightarrow q \) receives a true value.

The **Biconditional Implication** of statements \( p \) and \( q \) is the statement \( (p \rightarrow q) \land (q \rightarrow p) \). Its truth table is given by

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Let \( p \) and \( q \) be statements. The converse of the statement \( p \rightarrow q \) is the statement \( q \rightarrow p \). The inverse of \( p \rightarrow q \) is the statement \( \neg p \rightarrow \neg q \). The contrapositive form of \( p \rightarrow q \) is the statement \( \neg q \rightarrow \neg p \).

**Example** Let

\[
\begin{align*}
p & : \text{I get SARS.} \\
q & : \text{I should stay in hospital.}
\end{align*}
\]

The converse, inverse and contrapositive forms of \( p \rightarrow q \) are the following sentences

\[
\begin{align*}
q \rightarrow p & : \text{I should stay in hospital, then I get SARS.} \\
\neg p \rightarrow \neg q & : \text{If I didn’t get SARS, then I should not stay in hospital.} \\
\neg q \rightarrow \neg p & : \text{If I should not stay in hospital, then I didn’t get SARS.}
\end{align*}
\]

When considering a family of statements \( P(x) \) indexed by a variable \( x \) (such statement form indexed by a variable is called a predicate), we need universal quantifier to express whether all of the statements are simultaneously true or one of them is false, and existential quantifier to express whether at least one of the statements is true or all of them are false. The universal quantification of a predicate \( P(x) \) is the statement “for all values of \( x \) \( P(x) \) is true,” denoted \( \forall x \ P(x) \). This means that the statement \( \forall x \ P(x) \) has true value when all \( P(x) \) have true value and \( \exists x \ P(x) \) has false value when one of \( P(x) \) has false value. For example, let \( P(x) \) denote \( x + 1 < 4 \), where \( x \) are real numbers. Then \( \forall x \ P(x) \) is a false statement because \( P(4) \) is not a true statement. The existential quantification of a predicate \( P(x) \) is the statement “there exists a value of \( x \) for which \( P(x) \) is true,” denoted \( \exists x \ P(x) \). This means that \( \exists x \ P(x) \) has true value when there is at least one \( x \) such that \( P(x) \) has true value and \( \forall x \ P(x) \) has false value when all statements \( P(x) \) have false value. For example, let \( Q(x, y, z) \) denote \( x^2 + y^2 = z^2 \). Then \( \exists x \exists y \exists z \ Q(x, y, z) \) is a true statement, because \( Q(3, 4, 5) \) is true statement.

Note that the index \( x \) here is not a propositional variable and its values are sometimes specified by its domain \( X \). So we may have sentences “\( \forall x \in X, P(x) \)” and “\( \exists x \in X, P(x) \)”. For instance, let \( \sqrt{x} \) be the square root of real numbers \( x \) and let \( P(x) \) to denote the sentence that \( \sqrt{x} \) is irrational. The statement

\[ \text{“for all primes } x \text{ the number } \sqrt{x} \text{ is irrational”} \]

can be expressed as “\( \forall \text{primes } x, P(x) \)".

### 3 Tautology

A statement is called a tautology if it is always true for all possible values of its propositional variables; a contradiction if it is always false; and a contingency if it can be either true or false, depending on the truth values of its propositional variables. For instance, \( (p \rightarrow q) \lor \neg q \) is a tautology; \( (p \rightarrow q) \land p \land \neg q \) is a contradiction; and \( (p \rightarrow q) \lor \neg p \) is a contingency.

Two statements \( p \) and \( q \) are said to be logically equivalent or simply equivalent, written \( p \equiv q \) or even \( p = q \), if \( p \leftrightarrow q \) is a tautology; that is, \( p \) and \( q \) have the same truth values.

**Proposition 3.1** Let \( p, q, r \) be arbitrary statements. Then

(a) \( p \land q \equiv q \land p \)
(b) \( p \lor q \equiv q \lor p \)

c) \( p \land (q \land r) \equiv (p \land q) \land r \)

d) \( p \lor (q \lor r) \equiv (p \lor q) \lor r \)

e) \( p \land (q \lor r) \equiv (p \land q) \lor (p \land r) \)

(f) \( p \lor (q \land r) \equiv (p \lor q) \land (p \lor r) \)

g) \( p \land p \equiv p \)

(h) \( p \lor p \equiv p \)

(i) \( \neg(p) \equiv p \)

(j) \( \neg(p \land q) \equiv \neg p \lor \neg q \)

(k) \( \neg(p \lor q) \equiv \neg p \land \neg q \)

Example 1 \((p \rightarrow q) \leftrightarrow (\neg p) \lor q\) is a tautology.

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Example 2 \((p \rightarrow q) \leftrightarrow (\neg q) \rightarrow \neg p\) is a tautology.

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For instance, let us consider the statement

“If I get SARS, then I should stay in hospital.”

Let denote “I get SARS” and let \( q \) denote “I should stay in hospital.” One feel that statement “If I get SARS, then I should stay in hospital” is logically equivalent to the statements

“If I shouldn’t stay in hospital, then I don’t get SARS.”

It is also logically equivalent to

“I didn’t get SARS or I should stay in hospital.”

**Theorem 3.2**

(a) \( (p \rightarrow q) \equiv (\neg p) \lor q \)
(b) \((p \rightarrow q) \equiv (\neg q \rightarrow \neg p)\)

(c) \((p \leftrightarrow q) \equiv (p \rightarrow q) \wedge (q \rightarrow p)\)

**Theorem 3.3**

(a) \(\neg (\forall x \ P(x)) \equiv \exists x \neg P(x)\)

(b) \(\neg (\exists x \ P(x)) \equiv \forall x \neg P(x)\)

(c) \(\forall x \ (P(x) \wedge Q(x)) \equiv (\forall x \ P(x)) \wedge (\forall x \ Q(x))\)

(d) \(\exists x \ (P(x) \lor Q(x)) \equiv (\exists x \ P(x)) \lor (\exists x \ Q(x))\)

(e) \(\forall x \ P(x) \lor (\forall x \ Q(x)) \rightarrow \forall x \ (P(x) \lor Q(x))\) is a tautology.

(f) \(\exists x \ (P(x) \land Q(x)) \rightarrow (\exists x \ P(x)) \lor (\exists x \ Q(x))\) is a tautology.

(g) \((\exists x \ P(x)) \rightarrow (\forall x \ Q(x))) \rightarrow \forall x \ (P(x) \rightarrow Q(x))\) is a tautology.

(h) \(\exists x \ (P(x) \rightarrow Q(x)) \equiv (\forall x \ P(x)) \rightarrow (\exists x \ Q(x))\)

**Proof.** (a)-(d) are trivial.

(e) If the statement \(\forall x \ P(x)) \lor (\forall x \ Q(x))\) has \(T\) value, then \((\forall x \ P(x)) = T\) or \((\forall x \ Q(x)) = T\), say \((\forall x \ P(x)) = T\). Obviously, \((\forall x \ P(x)) \lor (\forall x \ Q(x)) = T\). Note that \((\forall x \ P(x)) \lor (\forall x \ Q(x))\) and \((\forall x \ P(x)) \lor (\forall x \ Q(x))\) are not equivalent.

(f) It is an equivalent form of (e).

(g) \((\exists x \ P(x)) \rightarrow (\forall x \ Q(x)) \equiv \neg (\exists x \ P(x)) \lor (\forall x \ Q(x)) \equiv (\forall x \ \neg P(x)) \lor (\forall x \ Q(x)); \forall x \ (P(x) \rightarrow Q(x)) \equiv (\forall x \ (\neg P(x)) \lor Q(x)).\) The tautology follows from (e).

(h) \((\exists x \ P(x) \rightarrow Q(x)) = (\exists x \ \neg P(x) \lor Q(x))\). It follows from (d) that \((\exists x \ \neg P(x) \lor Q(x))\) is equivalent to \((\exists x \ \neg P(x)) \lor (\exists x \ Q(x)) = \neg (\forall x \ P(x)) \lor (\exists x \ Q(x)) = (\forall x \ (x)) \rightarrow (\exists x \ Q(x)).\)

**Definition 3.4** A subset of connectives is called adequate if every statement can be represented by a statement form containing only connectives from that subset.

**Theorem 3.5** The subset \(\{\neg, \lor, \land\}\) is adequate. In this adequate subset, \(\lor\) can be replaced by either \(\land\) or \(\rightarrow\), and \(\land\) can be replaced by \(\lor\).

## 4 Methods of Proof

Let \(p\) and \(q\) be statements. If \(p \rightarrow q\) is a tautology, then we say that \(q\) follows logically from \(p\), and write \(p \Rightarrow q\). For statements \(p_1, p_2, \ldots, p_n\), if

\[
(P_1 \land P_2 \land \cdots \land P_n) \Rightarrow q,
\]

that is, \((p_1 \land p_2 \land \cdots \land p_n) \rightarrow q\) is a tautology, we say that \(q\) follows logically from \(p_1, p_2, \ldots, p_n\), and write

\[
\begin{array}{c}
P_1 \\
P_2 \\
\vdots \\
P_n \\
\hline
q
\end{array}
\]

5
The statements \( p_1, p_2, \cdots, p_n \) are called the hypothesis (or premises) and \( q \) the conclusion. To prove the theorem \( p \Rightarrow q \), it means to show that the implication \( p \Rightarrow q \) is a tautology. Arguments based on tautology are called rules of inference. The true of rules of inference is universal, and is independent of the context of the truth values of the simple statements involved.

**Modus Ponens**, also called **Rule of Detachment** (method of affirming), is the inference

\[
\begin{align*}
  p \\
  \hline
  p \Rightarrow q \\
  q
\end{align*}
\]

This means that the statement \((p \land (p \Rightarrow q)) \Rightarrow q\) is a tautology.

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<th>( p \Rightarrow q )</th>
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**Law of Syllogism**, also called **Chain Rule**, is the inference

\[
\begin{align*}
  p \Rightarrow q \\
  q \Rightarrow r \\
  \hline
  p \Rightarrow r
\end{align*}
\]

This means that the statement \(((p \Rightarrow q) \land (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)\) is a tautology. In fact, the statement has false value if and only if \((p \Rightarrow q) \land (q \Rightarrow r)\) has true value and \( p \Rightarrow r \) has false value. Then both \( p \Rightarrow q \) and \( q \Rightarrow r \) have true value, but \( p \) must have true value and \( r \) must have false value. Thus \( q \) must have true value by the true value of \( p \Rightarrow q \). Since \( q \Rightarrow r \) has true value, then \( r \) has true value, a contradiction.

**Proof by Contradiction**, also called **Modus Tollens** (method of denying), is the inference

\[
\begin{align*}
  p \Rightarrow q \\
  \neg q \\
  \hline
  \neg p
\end{align*}
\]

This means that the statement \(((p \Rightarrow q) \land \neg q) \Rightarrow \neg p\) is a tautology.

**Mathematical Induction** (MI) is the following inference about a family of statements \( P(k) \), indexed by positive integers \( k \)

\[
\begin{align*}
  P(1) \\
  \forall k (P(k) \Rightarrow P(k + 1)) \\
  \hline
  \forall k P(k)
\end{align*}
\]

Mathematical induction is a consequence of applying the Modus Ponens and the Law of Syllogism again and again.

**Example**
5 Boolean Functions

The truth values $T$ and $F$ are sometimes denoted by 1 and 0, and write $B = \{0, 1\}$. The $n$th product $B^n$ is called the $n$-dimensional binary space. Any function $f : B^n \to B$ is called a **Boolean function** of $n$ variables; the value of $f$ for Boolean variables $(x_1, \ldots, x_n) \in B^n$ is denoted by $f(x_1, \ldots, x_n)$; the variables $x_1, \ldots, x_n$ and their negations $\overline{x}_1, \ldots, \overline{x}_n$ are called **literals**. The variables $x_1, \ldots, x_n$ can be viewed as simple statements.

**Theorem 5.1** Any Boolean function can be written as a Boolean formula with literals and connectives of disjunction and conjunction.

**Proof.** We proceed by induction on the number of variables. For one variable $x$, there are four Boolean functions: $f_1(T) = f_1(F) = T$; $f_2(T) = f_2(F) = F$; $f_3(T) = T, f_3(F) = F$; $f_4(T) = F, f_4(F) = T$. It is clear that

$$f_1(x) = x \lor \overline{x}; \quad f_2(x) = x \land \overline{x}; \quad f_3(x) = x; \quad f_4(x) = \overline{x}.$$

Assume it is true for $n - 1$ variables; consider a Boolean function $f$ of $n$ variables $(x_1, \ldots, x_n)$. Define two Boolean functions of $n - 1$ variables as follows:

$$g_1(x_2, \ldots, x_n) = f(T, x_2, \ldots, x_n), \quad g_0(x_2, \ldots, x_n) = f(F, x_2, \ldots, x_n).$$

Then it is not hard to check

$$f(x_1, \ldots, x_n) = (x_1 \land g_1(x_2, \ldots, x_n)) \lor (\overline{x}_1 \land g_0(x_2, \ldots, x_n)).$$

By induction hypothesis, $g_1$ and $g_0$ can be written as Boolean formulas of literals and connectives of disjunction and conjunction; so does $f$. \qed

**Example** The Boolean function

\[
\begin{array}{c|c}
(x_1, x_2, x_3) & f(x_1, x_2, x_3) \\
\hline
(0,0,0) & 1 \\
(0,0,1) & 1 \\
(0,1,0) & 0 \\
(0,1,1) & 0 \\
(1,0,0) & 0 \\
(1,0,1) & 1 \\
(1,1,0) & 0 \\
(1,1,1) & 1
\end{array}
\]

can be written as

\[
f(x_1, x_2, x_3) = (x_1 \land g_1(x_2, x_3)) \lor (\overline{x}_1 \land g_0(x_2, x_3)),
\]

where

\[
g_1(x_2, x_3) = f(1, x_2, x_3) = (x_2 \land g_{11}(x_3)) \lor (\overline{x}_2 \land g_{10}(x_3)) = (x_2 \land x_3) \lor (\overline{x}_2 \land x_3) = x_3,
\]

\[
g_0(x_2, x_3) = f(0, x_2, x_3) = (x_2 \land g_{01}(x_3)) \lor (\overline{x}_2 \land g_{00}(x_3)) = (x_2 \land x_3 \land \overline{x}_3) \lor (\overline{x}_2 \land (x_3 \lor \overline{x}_3)) = \overline{x}_2.
\]
Thus
\[ f(x_1, x_2, x_3) = (x_1 \land x_3) \lor (\overline{x}_1 \land \overline{x}_2). \]

**Exercises**

1. Consider the statement

\[ \text{If } 1 = 4, \text{ then } 1 = 2. \]

*Proof.* Since \( 1 + 3 = 4 \) and \( 1 = 4 \), we have \( 0 = 3 \). Dividing both sides of \( 0 = 3 \) by 3, we further have \( 0 = 1 \). Hence \( 1 = 0 + 1 = 1 + 1 = 2 \). Is the proof a true argument? What can you conclude from the statement and proof?

2. Define the connectives “\(\downarrow\)” and “\(\Delta\)” by

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<th>( p \downarrow q )</th>
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respectively. Find the truth tables for

\( (p \downarrow q) \downarrow r, (p \downarrow q) \land (p \downarrow r), (p \downarrow q) \downarrow (p \downarrow r), (p \land q) \Delta p, (p \Delta q) \Delta (q \Delta r) \).

3. Let

\( p: \) John is a student of Computer Science Department in HKUST.

\( q: \) John takes course Math132

Write the English sentences of the converse, inverse and contrapositive forms for the statement \( p \rightarrow q \). Write the English sentence for the statements

\( \neg p \lor q, \neg q \lor p \)

4. Show that the set \( \{ \neg, \rightarrow, \exists \} \) is adequate. Are the sets \( \{ \neg, \downarrow, \forall \} \) and \( \{ \neg, \Delta, \exists \} \) adequate?

5. Show that the statement

\( (\forall x P(x)) \lor (\forall x Q(x)) \rightarrow \forall x (P(x) \lor Q(x)) \)

is a tautology. Is the converse of the statement a tautology? If yes, prove it. If no, find a counterexample.

6. Show that if statements \( p \) and \( p \rightarrow q \) are tautologies then \( q \) is a tautology. Give a daily life example of the argument.

7. Express \( p \downarrow q \) and \( p \Delta q \) in terms of \( p, q \), and other connectives without \( \downarrow \) and \( \Delta \).

8. If \( p \rightarrow q \) and \( q \rightarrow r \) are tautologies, then \( p \rightarrow r \) is a tautology.

9. If \( p \rightarrow q \) and \( \neg q \) are tautologies, then \( \neg p \) is a tautology.