Week 6: Fundamentals of Combinatorics (cont’d)

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1 Pigeonhole Principle

Theorem 1.1 (Pigeonhole Principle) If $n$ objects are placed into $m$ boxes and $n > m$, then there is at least one box which contains two or more objects.

The pigeonhole principle is a common Chinese saying: When pigeons are put into pigeonholes with more pigeons than pigeonholes, there are at least two pigeons must be put in a same pigeonhole.

Example Among any five integers between 1 and 8 inclusive, there are at least two of them adding up to 9.

Solution. We can divide the set \{1, 2, \cdots, 8\} into four disjoint subsets where each has two elements adding up to 9: \{1, 8\}, \{2, 7\}, \{3, 6\}, and \{4, 5\}. When selecting five numbers from these four subsets, at least two of the five selected numbers must come from a same subset of the four subsets. Thus their addition is 9.

Example Show that in any group of two or more persons there are at least two having the same number of friends (It is assumed that if a person $x$ is a friend of a person $y$ then $y$ is also a friend of $x$).

Solution. Assume that there are $n$ persons in the group. The number of friends of a person $x$ should be between 0 and $n - 1$. If there is one person $x^*$ who has $n - 1$ friends, then everyone is a friend of $x^*$. So both 0 and $n - 1$ can not be numbers of friends of some people in the group. Thus the pigeonhole principle tells us that there are at least two people who have the same number of friends.

Example Show that if $a_1, a_2, \cdots, a_k$ are integers (not necessarily distinct), then some of them can be added up to a multiple of $k$.

Solution. Consider the integers of the following $k + 1$ objects:

$$0, a_1, a_1 + a_2, a_1 + a_2 + a_3, \cdots, a_1 + a_2 + \cdots + a_k. \tag{1}$$

Note that integers modulo $k$ can only be 0, 1, 2, \ldots, $k - 1$. By the Pigeonhole Principle there are at least two integers in (1), say

$$a_1 + \cdots + a_i \quad \text{and} \quad a_1 + \cdots + a_j,$$
whose remainders modulo $k$ are the same. The number $a_1 + \cdots + a_i$ could be the 0, the very first element in (1). Thus 

$$a_{i+1} + a_{i+2} + \cdots + a_j$$

is a multiple of $k$.

**Example** Given 10 distinct integers $a_1, a_2, \ldots, a_{10}$ such that $0 \leq a_i < 100$, can we find a subset of $\{a_1, \ldots, a_{10}\}$ such that the sum of numbers in the subset with sign is zero?

**Solution.** Consider all possible partial sums of the selected numbers $a_1, a_2, \ldots, a_{10}$. The values of these sums should be between 0 and 1000. Note that the number of subsets of 10 objects is $2^{10} = 1024$. By the Pigeonhole Principle there are at least two subsets $A$ and $B$ of $\{a_1, a_2, \ldots, a_{10}\}$ such that the sum of the elements of $A$ and the sum of the elements of $B$ are the same, that is,

$$\sum_{a_i \in A} a_i = \sum_{a_j \in B} a_j.$$

Now we move all elements from the right side to the left; the elements in both $A$ and $B$ will be canceled. Thus sum of the elements of $A \Delta B$ with positive sign for the elements in $A - B$ and negative sign for the elements in $B - A$ is equal to 0.

**Theorem 1.2** If $n$ objects are placed in $m$ boxes, then one of the boxes must contain at least $\left\lceil \frac{n}{m} \right\rceil$ objects, where $\left\lceil \frac{n}{m} \right\rceil$ denotes the smallest integer greater or equal to $\frac{n}{m}$.

## 2 Relation to Probability

There are lot problems in our daily life about chance, the possibility or probability. When we flip a coin, we have two possible outcomes, Head and Tail. If the coin is fair, the chance to have the outcome – Head – is one-half or 50%. When we roll a pair of dice we may have outcomes – a collection of pairs of numbers between 1 and 6. The chance of the event of the outcomes that the sum of the pair is even is one-half. For instance, we may be interested in computing the probability of the event of the outcomes that the sum of the pair is 8.

**Definition 2.1** Any collection of outcomes in a probabilistic experiment is called an event. If each outcome is equally likely to be happened, we define

Probability of even $A = P(A) = \frac{\text{Total number of favorite outcomes}}{\text{Total number of possible outcomes}}$.

**Example** What is the probability of selecting three distinct numbers from 1, 2, \ldots, 11 so that two are less than 5, one is equal to 5, and four are larger than 5?

**Solution.** The total number of possible outcomes is $\binom{11}{3}$, and the total number of favorite outcomes is $\binom{4}{2} \binom{1}{1} \binom{6}{4}$. Then the probability is

$$\frac{\binom{4}{2} \binom{1}{1} \binom{6}{4}}{\binom{11}{3}}.$$

**Example** Find the probability that no two persons have the same birthday in a party of 40 people.
**Solution.** The total number of possible outcomes is $365^{40}$ and the total number of favorite outcomes is $\binom{365}{40}40!$. The probability is

$$\frac{\binom{365}{40}40!}{365^{40}} \approx 0.109.$$  

**Example** What is the probability of rolling a pair of dice so that the sum of numbers on the top faces is 8?

**Solution.** Since there is no order between the two dice, there are twenty-one possible outcomes

$$\{i, j\}, \ 1 \leq i \leq j \leq 6$$

and three favorite outcomes $\{2,6\}, \{3,5\}, \{4,4\}$. So the answer might be $\frac{3}{21} = \frac{1}{7}$. One may color the two dice as black and white so that the two dice are ordered. There are thirty-six ($= 6 \times 6$) possible outcomes and five favorite outcomes $(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)$. The answer should be $\frac{5}{36}$. Which one is correct and why?

**Example** Find the probability of rolling four dice simultaneously so that the sum of points is exactly 9.

**Solution.** The total number of possible outcomes is $6^4$. The total number of favorite outcomes is the number of positive integer solutions of the equation

$$x_1 + x_2 + x_3 + x_4 = 9$$

which is equivalent to the number of non-negative integer solutions of the equation

$$y_1 + y_2 + y_3 + y_4 = 5.$$  

Thus the answer is given by

$$\frac{\binom{4}{5}}{6^4} = \frac{7}{162} \approx \frac{1}{23}.$$  

**Exercises**

1. Show that there must be 90 ways to choose six numbers from 1 to 15 so that all the choices have the same sum.

2. Show that if five points are selected in a square whose sides have length 2, then there are at least two points whose distance is at most $\sqrt{2}$.

3. Prove that if any 14 numbers from 1 to 25 are chosen, then one of them is a multiple of another.

4. Twenty disks numbered 1 through 20 are placed face down on a table. Disks are selected one at a time and turned over until 10 disks have been chosen. If two of the disks add up to 21, the play loses. Is it possible to win this game?

5. Show that it is impossible to arrange the numbers $1, 2, \ldots, 10$ in a circle so that every triple of consecutively placed numbers has a sum less than 15.