1 Principles of Mathematical Induction

Consider the following three statements, each involving a general positive integer \( n \).

1. The sum of the first \( n \) odd numbers is equal to \( n^2 \).
2. If \( p \geq -1 \) then \((1 + p)^n \geq 1 + np\).
3. The sum of the internal angles of an \( n \)-sided convex polygon is \((n - 2)\pi\).

**Proposition 1** (Mathematical Induction). Suppose for each positive integer \( n \) we have a statement \( P(n) \). If we prove the following two things:

(a) (Induction Basis) \( P(1) \) is true;
(b) (Induction Hypothesis) If \( P(n) \) is true then \( P(n + 1) \) is also true;

Then \( P(n) \) is true for all positive integers \( n \).

The logic is clear: \( P(1) \Rightarrow P(2), P(2) \Rightarrow P(3), P(2) \Rightarrow P(3), \ldots \) It follows that \( P(n) \) is true for all positive integers \( n \).

**Example 1.** Let \( P(n) \) denote the statement: \( 1 + 3 + 5 + \cdots + (2n - 1) = n^2 \), where \( n \geq 1 \).

**Proof.**
(a) \( P(1) \) is true: \( 1 = 1^2 \).
(b) Suppose \( P(n) \) is true. Then \( 1 + 3 + 5 + \cdots + (2n - 1) = n^2 \). Adding \( 2n + 1 \) to both sides we have

\[
1 + 3 + 5 + \cdots + (2n - 1) + (2n + 1) = n^2 + 2n + 1 = (n + 1)^2.
\]

This means that the statement \( P(n + 1) \) is true. Thus \( P(n) \) is true for all positive integers \( n \).

**Example 2.** Let \( P(n) \) denote the statement: If \( p \geq -1 \) then \((1 + p)^n \geq 1 + np\) for all positive integers \( n \).

**Proof.**
(a) \( P(1) \) is true: \( 1 + p \geq 1 + p \).
(b) Suppose \( P(n) \) is true. Then \((1 + p)^n \geq 1 + np\). Since \( p \geq -1 \), then \( p + 1 \geq 0 \). Multiplying \( 1 + p \) to both sides, we have

\[
(1 + p)(1 + p)^n \geq (1 + p)(1 + np) = 1 + np + p + np^2 \geq 1 + (n + 1)p.
\]

This is exactly the statement \( P(n + 1) \). Thus \( P(n) \) is true for all positive integers \( n \).
**Proposition 2** (Mathematical Induction). Let $k$ be an integer. Suppose for each integer $n \geq k$ we have a statement $P(n)$. If we prove the following two things:

(a) (Induction Basis) $P(k)$ is true;

(b) (Induction Hypothesis) If $P(n)$ is true then $P(n+1)$ is also true;

Then $P(n)$ is true for all integers $n \geq k$.

**Example 3.** Let $P(n)$ denote the statement: The sum of internal angles of an $n$-sided convex polygon is $(n-2)\pi$.

Proof. Note that $n$ must be an integer larger than or equal to 3, i.e., $n \geq 3$.

(a) $P(3)$ is true: checked in junior high school.

(b) Suppose $P(n)$ is true. Let $A_1 A_2 \cdots A_{n+1}$ be an $(n+1)$-sided polygon whose vertices are $A_1, A_2, \ldots, A_{n+1}$. Draw a segment between the two vertices $A_1$ and $A_{n+1}$. We have a triangle $\Delta A_1 A_n A_{n+1}$ and an $n$-sided polygon $A_1 A_2 \ldots A_n$. Let $\alpha_i$ denote the internal angle of the polygon $A_1 A_2 \cdots A_{n+1}$ at the vertex $A_i$, $\beta_j$ the internal angle of the polygon $A_1 A_2 \cdots A_n$ at the vertex $A_j$, and $\gamma_k$ the internal angle of $\Delta A_1 A_n A_{n+1}$ at the vertex $A_k$. Then $\alpha_1 = \beta_1 + \gamma_1$, $\alpha_n = \beta_n + \gamma_n$, $\alpha_{n+1} = \gamma_{n+1}$, and $\alpha_i = \beta_i$ ($2 \leq i \leq n-1$). Thus

\[
\alpha_1 + \alpha_2 + \cdots + \alpha_{n+1} = (\beta_1 + \gamma_1) + \beta_2 + \cdots + \beta_{n-1} + (\beta_n + \gamma_n) + \gamma_{n+1} = (\gamma_1 + \gamma_n + \gamma_{n+1}) + (\beta_1 + \cdots + \beta_{n-1} + \beta_n) = \pi + (n-2)\pi = (n-1)\pi = ((n+1) - 2)\pi.
\]

This means that the statement $P(n+1)$ is true. So $P(n)$ is true for all integers $n \geq 3$.

**Example 4.** Let $P(n)$ denote the statement: $2^n < n!$, where $n \geq 1$.

Proof. (a) $P(1)$ is true: Trivial.

(b) Suppose $P(n)$ is true, i.e., $2^n < n!$. Then

\[
2^{n+1} = 2 \cdot 2^n < (n+1) \cdot n! = (n+1)!.
\]

This is exactly the statement $P(n+1)$. Thus $P(n)$ is true for all positive integers $n$. What is wrong? □

Consider the problem: $2^n \leq n!$ for $n \geq 0$.

**Proposition 3** (Strong Mathematical Induction). Let $k$ be an integer. Suppose for each integer $n \geq k$ we have a statement $P(n)$. If we prove the following two things:

(a) (Induction Basis) $P(k)$ is true.

(b) (Strong Induction Hypothesis) If $P(k), P(k+1), \ldots, P(n)$ are true then $P(n+1)$ is also true.

Then $P(n)$ is true for all integers $n \geq k$.

**Example 5.** Let $P(n)$ denote the statement: $u_n = 2^n + 1$, where $u_n$ is the sequence with $n \geq 0$, $u_0 = 2$, $u_1 = 3$, and

\[
u_{n+1} = 3u_n - 2u_{n-1}, \quad n \geq 1.
\]
Proof. (a) \( P(1) \) is true: \( 2 = 1 + 1 = 2^0 + 1 \).
(b) Suppose \( P(n) \) is true, i.e., \( u_n = 2^n + 1 \). Then by induction hypothesis
\[
\begin{align*}
u_{n+1} &= 3u_n - 2u_{n-1} = 3 \cdot (2^n + 1) - 2 \cdot (2^{n-1} + 1) = 2^{n+1} + 1.
\end{align*}
\]
This means that the statement \( P(n+1) \) is true. Thus \( P(n) \) is true for all integers \( n \geq 0 \).

Example 6. Find a closed formula for the sum \( 1^2 + 2^2 + \cdots + n^2 \).

Proof. It is known that \( 1 + 2 + \cdots + n = \frac{1}{2}n(n+1) \). It suggests that the wanted closed formula for the sum is a polynomial of degree 3. Set
\[
1^2 + 2^2 + \cdots + n^2 = a_0 + a_1 n + a_2 n^2 + a_3 n^3.
\]
Then for \( n = 1, 2, 3, 4 \), we have the system of linear equations
\[
\begin{align*}
a_0 &+ a_1 &+ a_2 &+ a_3 = 1 \\
a_0 &+ 2a_1 &+ 2^2a_2 &+ 2^3a_3 = 5 \\
a_0 &+ 3a_1 &+ 3^2a_2 &+ 3^3a_3 = 14 \\
a_0 &+ 4a_1 &+ 4^2a_2 &+ 4^3a_3 = 30
\end{align*}
\]
By the Cramer’s rule we have that the unique solution: \( a_0 = 0, a_1 = \frac{1}{6}, a_2 = \frac{1}{2}, a_3 = \frac{1}{3} \). Now we prove by mathematical induction the proposed formula
\[
P(n) : 1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1).
\]
(a) \( P(1) \) is true: \( 1 = \frac{1}{6} \cdot 2 \cdot 3 \).
(b) Suppose \( P(n) \) is true. Then by induction hypothesis
\[
\begin{align*}
1^2 + 2^2 + \cdots + n^2 + (n+1)^2 &= \frac{1}{6}n(n+1)(2n+1) + (n+1)^2 \\
&= \frac{1}{6}(n+1)[n(2n+1) + 6(n+1)] \\
&= \frac{1}{6}(n+1)(2n^2 + 7n + 6) \\
&= \frac{1}{6}(n+1)(n+2)(2n+3) \\
&= \frac{1}{6}(n+1)[(n+1) + 2][2(n+1) + 3].
\end{align*}
\]
This means that the statement \( P(n+1) \) is true. Thus \( P(n) \) is true for all integers \( n \geq 1 \).

Definition 4. A prime number is a positive integer \( p \) such that \( p \geq 2 \) and the only positive integers dividing \( p \) are 1 and \( p \).

Proposition 5. Every positive integer greater than 1 is equal to a product of prime numbers.

Proof. For each positive integer \( n \geq 2 \), let \( P(n) \) denote the statement: the integer \( n \) is equal to a product of prime numbers.
(a) \( P(2) \) is true, since 2 is a prime number.
(b) \( P(2), P(3), \ldots, P(n) \Rightarrow P(n+1) \): For the positive integer \( n+1 \), if \( n+1 \) is a prime \( p \), then \( n + 1 = p \) is already a product of prime numbers (only one prime number in the product); if \( n+1 \)
is not a prime number, then there is a positive integer \( a \) dividing \( n + 1 \). Writing \( b = \frac{n+1}{a} \), we have \( b \) is an integer, \( b \geq 2 \), and

\[ n + 1 = ab, \quad \text{where} \ a, b \in \{2, 3, \ldots, n\}. \]

By induction hypothesis, the positive integers \( a \) and \( b \) have prime factorizations, say, \( a = p_1p_2 \cdots p_k \) and \( b = q_1q_2 \cdots q_l \). Then \( n + 1 = p_1p_2 \cdots p_kq_1q_2 \cdots q_l \). This means that \( n + 1 \) is a product of prime numbers.

\[ \Box \]

2 Euler’s Formula and Platonic Solids

A polyhedron is a solid whose surface consists of a number of faces, all of which are convex polygons, such that any side lies on exactly one other face. The corners of the faces are called vertices of the polyhedron, and their sides are the edges of the polyhedron.

A polyhedron is said to be convex if, whenever we choose two points on its surface, the straight line joining them lies entirely within the polyhedron.

Example 7. (a) A cube has 8 vertices, 12 edges, and 6 faces.
(b) A tetrahedron has 4 vertices, 6 edges, and 4 faces.
(c) The prism whose base is a rectangle is a polyhedron, having 5 vertices, 8 edges, and 5 faces.

Theorem 6 (Euler’s Formula). For any convex polyhedron with \( V \) vertices, \( E \) edges, and \( F \) faces, we have the relation

\[ V - E + F = 2. \]

Proof. It follows from the following theorem.

Definition 7. A graph is a figure in the plane consisting of a collection of points (called vertices) and some edges joining various pairs of these points. A graph is connected if we can get from any vertex of the graph to any other vertex by going along a path of edges in the graph. A graph is called a plane graph if there are no two edges crossing each other. A loop in a graph is an edge that joins two identical vertices.

A plane graph separate the plane into some connected regions.

Theorem 8 (Euler’s Relation). For any connected plane graph \( G \) with \( v \) vertices, \( e \) edges, and \( r \) regions, we have

\[ v - e + r = 2. \]

Proof. Let \( P(n) \) be the statement: every connected plane graph with \( n \) edges satisfies the formula \( v - e + r = 2 \). Note that \( P(n) \) is a statement about lots of plane graphs. For instance, \( P(1) \) is a statement about two plane graphs: a segment graph and a loop graph. We apply mathematical induction to prove \( P(n) \) for all positive integers \( n \).

(a) \( P(1) \) is true: A segment graph has 2 vertices, 1 edge, and 1 face; so \( 2 - 1 + 1 = 2 \). A loop graph has 1 vertex, 1 edge, and 2 faces; so \( 1 - 1 + 2 = 2 \).

(b) Suppose \( P(n) \) is true for \( n \), i.e., every plane graph with \( n \) edges satisfies the Euler formula. Let \( G \) be a plane graph with \( n + 1 \) edges. We have two cases.

Case 1: \( G \) has a bounded region. Let \( x \) be an edge of \( G \) bounding a bounded region; and let \( G' \) be a graph obtained from \( G \) by removing the edge \( x \). It is clear that \( G' \) is connected and planar, \( v(G) = v(G'), e(G) = e(G') + 1, \) and \( r(G) = r(G') + 1 \). Since \( v(G') - e(G') + r(G') = 2 \), we have

\[ v(G) - e(G) + r(G) = v(G') - [e(G') + 1] + [r(G') + 1] = v(G') - e(G') + r(G') = 2. \]
This means that $P(n+1)$ is true.

**Case 2:** $G$ has no bounded region. Then $G$ has the only unbounded region. So $G$ has no closed path. It follows that $G$ has an end-vertex, a vertex joined by only one edge. Otherwise, if each vertex is joined by two edges, then we can start to travel on edges to obtain a closed path from one vertex, reaching an vertex through an edge and leaving the same vertex through another edge.]

Take a end-vertex of $G$ and remove the end-vertex and the only edge joining to it; we obtain a connected plane graph $G'$. Note that $v(G) = v(G') + 1$, $e(G) = e(G') + 1$, and $r(G) = r(G')$. then

$$v(G) - e(G) + r(G) = [v(G') + 1] - [e(G') + 1] + r(G') = v(G') - e(G') + r(G') = 2.$$ 

Now we have seen that $P(n+1)$ is true. The proof is finished.

3 Regular and Platonic Solids

A convex polygon is said to be **regular** if all its sides are of equal length and all its internal angles are equal. A polyhedron is said to be **regular** if (i) all its faces (convex polygons) are regular and have the same number of sides; (ii) all vertices have the same number of edges joining them. The Platonic solids are the five regular polyhedra: cube, tetrahedron, octahedron, dodecahedron, and icosahedron.

**Theorem 9.** The only regular convex polyhedra are the five Platonic solids.

**Proof.** Let $P$ be a regular polyhedron with $v$ vertices, $e$ edges, and $f$ faces. Let $n$ be the number of sides of a face, and $d$ the number of edges joining a vertex. Then

$$2e = nf,$$

[It follows from the counting of the number of ordered pairs $(\varepsilon, \sigma)$, where $\varepsilon$ is an edge, $\sigma$ is a face, and $\varepsilon$ bounds $\sigma$.]

$$2e = dv.$$ 

[It follows from the counting of the number of ordered pairs $(\nu, \varepsilon)$, where $\nu$ is a vertex, $\varepsilon$ is an edge, and $\varepsilon$ joins $\nu$.] Thus

$$f = \frac{2e}{n}, \quad v = \frac{2e}{d}.$$ 

Recall the Euler formula $v - e + f = 2$; we have $\frac{2e}{d} - e + \frac{2e}{n} = 2$. Dividing both sides by $2e$, we have

$$\frac{1}{d} + \frac{1}{n} = \frac{1}{e} + \frac{1}{2}. \quad (1)$$

Note that $n \geq 3$, as a convex polygon must have at least 3 sides; likewise $d \geq 3$, since it is geometrically clear that in a polyhedron a vertex must belong to at least 3 edges. Since the right hand side of (1) is at least $\frac{1}{2}$, it follows that we cannot have both $d \geq 4$ and $n \geq 4$. So we have either $d \leq 3$ or $n \leq 3$, and subsequently either $d = 3$ or $n = 3$.

**Case $d = 3$.** Then (1) becomes

$$\frac{1}{n} = \frac{1}{e} + \frac{1}{6}.$$ 

Since $e$ is positive, it follows that $3 \leq n \leq 5$. So $(n, e) = (3, 6), (4, 12), (5, 30)$; i.e., $(v, e, f) = (4, 6, 4), (8, 12, 6), (20, 30, 12)$.

**Case $n = 3$.** Then (1) becomes

$$\frac{1}{d} = \frac{1}{e} + \frac{1}{6}.$$
Since $e$ is positive, it follows that $3 \leq d \leq 5$. So $(d, e) = (3, 6), (4, 12), (5, 30)$; i.e., $(v, e, f) = (4, 6, 4), (6, 12, 8), (12, 30, 20)$.

We thus have five regular polyhedra: tetrahedron $(4, 6, 4)$; cube $(8, 12, 6)$; octahedron $(6, 12, 8)$; dodecahedron $(20, 30, 12)$; icosahedron $(12, 30, 20)$.

A complete graph $K_n$ is a graph with $n$ vertices such that every two vertices are adjacent by an edge. The complete graph $K_5$ is not planar. Since $v(K_5) = 5, e(K_5) = 10$, if $K_5$ is planar then by the Euler formula we have $f(K_5) = 2 - v + e = 7$, i.e., $K_5$ has 7 faces. Since $2e \geq 3f$, it follows that $20 = 2e \geq 3f = 21$, this is a contradiction.

A complete bipartite graph is a graph $K_{m,n}$ whose vertex set can be divided into two parts $V_1$ and $V_2$ with $|V_1| = m$ and $|V_2| = n$, and the edges set is $V_1 \times V_2$. The complete bipartite graph $K_{3,3}$ is non-planar. Note that $v = 6$ and $e = 9$. If $K_{3,3}$ is planar, then by the Euler formula we have $r = 2 - v + e = 5$ regions. Note that every cycle of a bipartite graph has even length, so every cycle of $K_{3,3}$ has length at least 4. Thus $2e \geq 4f$ implies $18 \geq 20$. This is a contradiction.

**Example 8.** The football graph has faces of pentagons and hexagons. Every vertex shares 3 edges and every edge shares 2 vertices. Each pentagon is surrounded by 5 hexagons and each hexagon is surrounded by 3 pentagons. Find the number of vertices, edges, pentagons, and hexagons of the football graph.

Let $v, e$ be the number of vertices and edges respectively. Let $f_5, f_6$ be the number of pentagons and hexagons. Then

$$3v = 2e, \quad 5f_5 + 6f_6 = 2e, \quad v - e + f_5 + f_6 = 2.$$ 

Note that the number of edges shared by both pentagons and hexagons is counted in two ways: counting by pentagons, counting by hexagons. We then have $5f_5 = 3f_6$.

Put $e = 3v/2$ into other equations, we have

$$v - \frac{3}{2}v + f_5 + f_6 = 2, \quad 5f_5 + 6f_6 = 3v, \quad 5f_5 = 3f_6.$$ 

Thus $f_5 = \frac{3}{5}f_6$.

$$-\frac{1}{2}v + \frac{3}{5}f_6 + f_6 = 2, \quad 3f_6 + 6f_6 = 3v.$$ 

$$v = 3f_6, \quad -5v + 16f_6 = 20.$$ 

$$f_6 = 20, \quad v = 60, \quad e = 90, \quad f_5 = 12.$$