1 Vertex Connectivity

Let $G = (V, E)$ be a graph, $x, y \in V$.

- Tow $(x, y)$-path $P$ and $Q$ are said to be **internally disjoint** if they have no internal vertices in common.

- The **local connectivity** between two distinct vertices $x$ and $y$ is the maximum number of pairwise internally disjoint $xy$-paths, denoted $p(x, y)$ or $p_G(x, y)$.

- A nontrivial graph $G$ is said to be $k$-**connected** if $p(u, v) \geq k$ for any two distinct vertices $u, v$. The **connectivity** of $G$ is the maximum value of $k$ for which $G$ is $k$-connected.

- A trivial graph (i.e. a graph with a single vertex and no edges) is 0-connected and 1-connected, but not 2-connected.

- The complete graph $K_n$ has $n - 2$ internally disjoint paths of length 2 and one path of length 1. So the connectivity of $K_n$ is $n - 1$.

- Let $G$ be a complete graph with multiple edges. Let $\mu(x, y)$ be the number edges between $x$ and $y$. Then there $\mu(x, y)$ $(x, y)$-paths of length 1 and $n - 2$ internally disjoint $(x, y)$-paths. So the local connectivity between $x$ and $y$ is $n - 2 + \mu(x, y)$.

- Let $\mu$ be the minimum number of multiple edge between two distinct vertices of a complete graph $G$ with multiple edges. Then the local connectivity $G$ is $n - 2 + \mu$.

- let $x, y$ be two vertices nonadjacent in a graph $G$. An $(x, y)$-**vertex-cut** is a subset $S \subseteq V - \{x, y\}$ such that $x, y$ belong to different components of $G - S$; we say that such a cut separates $x$ and $y$. We denote by $c(x, y)$ the minimum size of an $(x, y)$-vertex-cut.

- A vertex cut of graph $G$ is an $(x, y)$-vertex-cut for at least one pair of $(x, y)$ of nonadjacent vertices. A vertex cut with $k$ vertices is referred to a $k$-**vertex cut**.

**Theorem 1.1** (Menger’s Theorem). Let $G(x, y)$ be a graph with two nonadjacent vertices $x, y$. Then the maximum number of pairwise internally disjoint $(x, y)$-paths is equal to the minimum number of vertices in an $(x, y)$-vertex-cut, i.e.,

\[ p(x, y) = c(x, y). \]

**Proof.** Set $p := p_G(x, y)$, $k := c_G(x, y)$. There are $p$ internally disjoint $(x, y)$-paths, and a vertex $k$-subset $K \subseteq V - \{x, y\}$ that separates $x$ and $y$. Since every $(x, y)$-path meets $S$ at an internal vertex, the $p$ internally disjoint $(x, y)$-paths meet $S$ at $p$ vertices. Hence $p_G(x, y) \leq c_G(x, y)$. To prove $p_G(x, y) \geq c_G(x, y)$, we proceed by induction on the number of edges of $G$. We may assume that there is an edge $e$ whose end-vertex is neither $x$ nor $y$; otherwise, every $(x, y)$-path is of length 2, and the conclusion is obviously true.
Set $H := G \setminus e$. Since $|E(H)| < |E(G)|$ and $p_H(x, y) \leq c_H(x, y)$, we have $p_H(x, y) = c_H(x, y)$ by induction. Moreover, $c_G(x, y) \leq c_H(x, y) + 1$, since any $(x, y)$-vertex-cut of $H$, together with an end-vertex of $e$, is an $(x, y)$-vertex-cut of $G$. Hence

$$p_G(x, y) \geq p_H(x, y) = c_H(x, y) \geq c_G(x, y) - 1 = k - 1.$$ 

If $p_G(x, y) = k$, then there is nothing to prove. So we may assume that $p_G(x, y) = p_H(x, y) = c_H(x, y) = k - 1$ and $c_G(x, y) = k$. Let $S := \{v_1, \ldots, v_{k-1}\}$ be a minimum $(x, y)$-vertex-cut of $H$. Let $X$ be the set of vertices reachable from $x$ in $H - S$, and $Y$ the set of vertices reachable from $y$ in $H - S$. Since $|S| = k - 1$, the set $S$ is not an $(x, y)$-vertex-cut of $G$; so there is an $(x, y)$-path in $G - S$. This path necessarily contains the edge $e$, and $e$ must have end-vertices $u \in X$ and $v \in Y$.

Now consider the graph $G/Y$ by contracting $Y$ to $y$. It is clear that every $(x, y)$-vertex-cut in $G/Y$ is an $(x, y)$-vertex-cut in $G$. Thus $c_{G/Y}(x, y) \geq k$. Note that $c_{G/Y}(x, y) \leq k$, because $S \cup \{u\}$ is an $(x, y)$-vertex-cut of $G/Y$. So $c_{G/Y}(x, y) = k$. Since $|E(G/Y)| < |E(G)|$, by induction there are $k$ internally disjoint $(x, y)$-paths $P_1, \ldots, P_k$ in $G/Y$, and each vertex of $S \cup \{u\}$ lies on one of them. Without loss of generality, we may assume that $v_i \in P_i$, $1 \leq i \leq k - 1$, and $u \in P_k$. Likewise, there are $k$ internally disjoint $(x, y)$-paths $Q_1, \ldots, Q_k$ in $G/X$ such that $v_i \in Q_i$, $1 \leq i \leq k - 1$, and $v \in Q_k$. Then there are $k$ internally disjoint $(x, y)$-paths $P_iQ_i$ ($1 \leq i \leq k - 1$) and $P_k'eQ_k'$ in $G$, where $P_k = P'e, Q_k = Q_k'e$. 

\[\square\]