Week 1-2: Graphs and Subgraphs

February 7, 2010

1 Graphs

Definition of Graphs:

- A graph $G$ is an ordered pair $(V, E)$ consisting of a set $V$ of vertices and a set $E$ (disjoint from $V$) of edges, together with an incidence function $\text{End} : E \to M_2(V)$, where $M_2(V)$ is the set of all 2-element sub-multisets of $V$; we usually write $V = V(G)$, $E = E(G)$, and $\text{End} = \text{End}_G$.

- If $x$ is an edge and $u, v$ are vertices such that $\text{End}(x) = \{u, v\}$, we say that $x$ joins $u, v$ (or $u, v$ are incident with $x$ or $u, v$ are adjacent by $x$), and $u, v$ are called end-vertices of $x$; we say that $x$ is a link if $u \neq v$ and a loop if $u = v$.

- Two edges are said to be parallel if their ends are the same.

Simples Graphs, Multigraphs, Complete Graphs, Bipartite Graphs:

- A graph is said to be simple if it has no loops and parallel edges. A graph with possible loops and parallel edges is also called a multigraph.

- A graph is said to be finite if its both vertex set and edge set are finite and assume all graphs are finite.

- The graph with empty vertex set (and hence empty edge set) is called the null graph.

- A graph is said to be trivial if it has only one vertex. All other graphs are said to be nontrivial.

- A graph is called an empty graph if it does not contain any edge.

- A complete graph is a simple that every pair of vertices are adjacent. A complete graph with $n$ vertices is denoted by $K_n$.

- A graph $G$ is said to be bipartite if its vertex set $V(G)$ can be partitioned into two disjoint parts $X$ and $Y$ such that every edge has one end-vertex in $X$ and one in $Y$; such a partition $\{X, Y\}$ is called a bipartition of $G$, and such bipartite graph is denoted by $G[X, Y]$.

- A bipartite graph $G[X, Y]$ is called a complete bipartite graph if every vertex in $X$ is joined to every vertex in $Y$; we denote $G[X, Y]$ by $K_{m,n}$ if $|X| = m$ and $|Y| = n$.

Neighbors, Degree:

- Two adjacent vertices are also called neighbors. The set of neighbors of a vertex $v$ in a graph $G$ is the set of all neighbors of $v$, denoted $N_G(v)$.

- The degree of a vertex $v$ in a graph $G$, denoted by $d_G(v)$, is the number of edges incident with the vertex, where loops are counted twice. A vertex is said to be isolated if its degree is 0. For a simple graph, $d_G(v) = |N_G(v)|$. 

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• For any graph $G = (V, E)$,
  \[ 2|E| = \sum_{v \in V} d_G(v). \]

• In any graph, the number of vertices of odd degree is even.

**Proposition 1.1.** Let $G[X, Y]$ be a bipartite graph without isolated vertices and $d(x) \geq d(y)$ for all edges $xy$ with $x \in X$ and $y \in Y$. Then $|X| \leq |Y|$, and the equality holds if and only if $d(x) = d(y)$ for all edges $xy$ with $x \in X$ and $y \in Y$.

**Proof.** Since $d(x) \geq d(y)$ for all edges $xy$ with $x \in X$ and $y \in Y$, we have
\[
|X| = \sum_{x \in X} \sum_{y \in Y} \frac{1}{d(x)} \leq \sum_{x \in X} \sum_{y \in Y} \frac{1}{d(y)} = |Y|.
\]
It is clear that if $d(x) = d(y)$ for all $xy \in E$ with $x \in X$ and $y \in Y$ then $|X| = |Y|$. Conversely, if $|X| = |Y|$, the above middle inequality must be equality. It forces that $d(x) = d(y)$ for all $xy \in E$ with $x \in X$ and $y \in Y$. \qed

**Incidence Matrix, Adjacency Matrix:**

• The **incidence matrix** of a graph $G$ is a matrix $M = M_G$, whose rows are indexed by vertices and whose columns are indexed by edges of $G$, such that (i) the entry $m_{ve} = 0$ at $(v, e)$ if the vertex $v$ is not incident with the edge $e$, (ii) $m_{ve} = 1$ if $v$ is incident with $e$ once (i.e., $e$ is a link), and (iii) $m_{ve} = 2$ if $v$ is incident with $e$ twice (i.e., $e$ is a loop).

• The **adjacency matrix** of a graph $G$ is a square matrix $A = A_G$, whose rows and columns are indexed by vertices of $G$, such that (i) the entry $a_{uv} = 0$ at $(u, v)$ if the vertices $u$ and $v$ are not adjacent, (ii) $a_{uv} = 1$ if $u$ and $v$ are adjacent by a link, and (iii) $a_{uv} = 2$ if $u$ and $v$ are adjacent by a loop ($u$ and $v$ must be identical).

**Walks, Trails, Paths, Cycles, Connectedness:**

• A **walk** from a vertex $u$ to a vertex $v$ in a graph $G$ is a sequence $W := v_0e_1v_1 \ldots v_{\ell-1}e_\ell v_\ell$ with $v_0 = u$ and $v_\ell = v$, whose terms alternate between vertices and edges of $G$, such that the edge $e_i$ is incident with the vertices $v_{i-1}$ and $v_i$, $1 \leq i \leq \ell$. The vertex $v_0$ is called the **initial vertex**, $v_\ell$ the **terminal vertex**, and the number $\ell$ the **length** of $W$. A walk is said to be **closed** if its initial and terminal vertices are identical.

• A walk is called a **trail** if its edge terms are distinct.

• A walk is called a **path** if its vertex and edge terms are distinct, except possible identical initial and terminal vertices. If $P = v_0e_1v_1 \ldots v_{\ell-1}e_\ell v_\ell$ is a path, then $v_0, v_1, \ldots , v_\ell$ are distinct or $v_0 = v_\ell$, $v_1, \ldots , v_{\ell-1}$ are distinct, and $v_1, \ldots , v_{\ell-1}$ are called **internal vertices** of $P$. A closed path is usually called a **cycle**.

• A graph is said to be **connected** if there is a path between any two vertices of the graph.

• An **Euler trail** of a graph $G$ is a trail that uses every edge of $G$. A closed Euler trail is called an **Euler tour**.

• A **Hamilton path** of a graph $G$ is a path that uses every vertex of $G$. A closed Hamilton path is called a **Hamilton cycle**.

**Union, Intersection, Cartesian Product:**

• Two graphs are said to be **disjoint** if they have no vertex in common, and **edge-disjoint** if they have no edge in common.

• The **union** of two graphs $G$ and $H$ is the graph $G \cup H$ with the vertex set $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H)$. If $G$ and $H$ are disjoint, we write their union as $G + H$.

• The **intersection** of two graphs $G$ and $H$ is the graph $G \cap H$ with the vertex set $V(G) \cap V(H)$ and the edge set $E(G) \cap E(H)$. note that if $G$ and $H$ are disjoint, then $G \cap H$ is the null graph.
• The **cartesian product** of simple graphs $G$ and $H$ is the graph $G \square H$ whose vertex set is $V(G) \times V(H)$ and whose edge set is

$$\{(u, x)(v, x) \mid uv \in E(G)\} \cup \{(u, x)(u, y) \mid xy \in E(H)\}.$$ 

Digraphs, Out-Degree, In-Degree, Orientations:

• A **directed graph** (or **digraph**) is an ordered pair $D = (V, A)$ consisting of a set $V$ of **vertices** and a set $A$ of **arcs**, together with an **incidence function** $\text{End} : A \to V \times V$. If $a$ is an arc and $\text{End}(a) = (u, v)$, we call the arc $a$ a **directed edge** from $u$ to $v$, the vertex $u$ a **tail**, and $v$ a **head** of $a$. We usually write $V = V(D)$, $A = A(D)$, and $\text{End} = \text{End}_D$.

• Let $v$ be a vertex in a digraph $D$. The **out-degree** of $v$ is the number of arcs of which $v$ is a tail, denoted $d_D^-(v)$. The **in-degree** of $v$ is the number of arcs of which $v$ is a head, denoted $d_D^+(v)$.

• Let $a$ be an arc in a digraph $D$ such that $\text{End}_D(a) = (u, v)$. We call $u$ an **in-neighbor** of $v$, and $v$ an **out-neighbor** of $u$. We denote by $N_D^-(v)$ the set of all out-neighbors of a vertex $v$, and by $N_D^+(v)$ the set of all in-neighbors of $v$.

• An **orientation** of an edge $e$ incident with two vertices $u, v$ in a graph $G$ is an assignment of signs to the pairs $(u, e)$ and $(v, e)$ such that their product is negative. A link edge has exactly two orientations. A loop has only one orientation.

• An **orientation** on a graph $G$ is an assignment that each edge is given an orientation. An orientation of $G$ can be considered as a function $\varepsilon : V \times E \to \{-1, 0, 1\}$ such that (i) $\varepsilon(v, e) = 0$ if the vertex $v$ is not incident with the edge $e$, (ii) $\varepsilon(u, e)\varepsilon(v, e) = -1$ if the edge $e$ joins the vertices $u$ and $v$. A graph $G$ together with an orientation $\varepsilon$ is called an **oriented graph**, denoted $(G, \varepsilon)$.

• An oriented graph $(G, \varepsilon)$ can be considered as a digraph $D$ with the vertex set $V(G)$, where each edge $e \in E$ incident with vertices $u, v$ is a directed edge from $u$ to $v$ if $\varepsilon(u, e) = +1$ and $\varepsilon(v, e) = -1$. Conversely, a digraph $D$ can be considered as an oriented graph $(G, \varepsilon)$ with the vertex set $V(D)$, where each directed edge $e$ from a vertex $u$ to a vertex $v$ is oriented by $\varepsilon(u, e) = +1$ and $\varepsilon(v, e) = -1$.

• A directed complete graph is called a **tournament**.

**Theorem 1.2.** Every tournament has a directed Hamilton path.

**Proof.** Let $D$ be a tournament with $n$ vertices. We proceed by induction on $n$. For $n = 2, 3$, it is easy to check directly. Now remove one vertex $v$ from $D$ to obtain a tournament $D - v$ with $n - 1$ vertices. By induction hypothesis, $D - v$ has a directed Hamilton path $P = v_1v_2 \cdots v_{n-1}$ from $v_1$ to $v_{n-1}$. The situation can be divided into the following cases.

**Case 1.** $(v, v_1)$ is a directed edge in $D$. Then $P_1 := vv_1v_2 \cdots v_{n-1}$ is a directed Hamilton path of $D$.

**Case 2.** $(v_1, v)$ and $(v, v_2)$ are directed edges in $D$. Then $P_2 := v_1v_2v_3 \cdots v_{n-1}$ is a directed Hamilton path of $D$.

**Case 3.** $(v_1, v), (v_2, v)$, and $(v, v_3)$ are directed edges in $D$. Then $P_3 := v_1v_2v_3 \cdots v_{n-1}$ is a directed Hamilton path of $D$. In general,

**Case k.** $(v_1, v), (v_2, v), \ldots, (v_{k-1}, v), (v, v_k)$ are directed edges in $D$, where $1 \leq k \leq n - 1$. Then

$$P_k := v_1v_2 \cdots v_{k-1}vv_k \cdots v_{n-1}$$

is a directed Hamilton path of $D$.

**Case n.** $(v_1, v), (v_2, v), \ldots, (v_{n-1}, v)$ are directed edges in $D$. Then $P_n := v_1v_2 \cdots v_{n-1}$ is a directed Hamilton path of $D$. \(\square\)

**Isomorphism, Automorphism:**

• Two graphs $G$ and $H$ are said to be **identical** if $V(G) = V(H)$ and $E(G) = E(H)$.

• A graph $G$ is said to be **isomorphic** to a graph $H$ if there are bijections $f : V(G) \to V(H)$ and $g : E(G) \to E(H)$ such that $\text{End}_G(e) = uv$ if and only if $\text{End}_H(g(e)) = f(u)f(v)$; such a pair $(f, g)$ of mappings is called an **isomorphism** from $G$ to $H$. 

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An isomorphism from a graph $G$ to itself is called an **automorphism** of $G$. The set of all automorphisms of $G$ forms a group under the composition of mappings, called the **automorphism group** of $G$, denoted $\text{Aut}(G)$.

**Labelled Graphs**

- Let $V$ be a finite set. A simple graph $G = (V, E)$ on $V$ can be considered as a subset of $\binom{V}{2}$, the set of all 2-element subsets of $V$. A simple graph whose vertices are labeled, but whose edges are not labeled, is referred as a **labeled simple graph**.

- For a set $V$ of $n$ elements, there are $2^\binom{n}{2}$ labeled simple graphs with vertex set $V$. We denote by $G(V)$ the set of all labeled simple graphs with vertex set $V$.

- Let $G$ be an unlabeled graph with $n$ vertices. Then the number of labelings of $G$ is $\frac{n!}{|\text{Aut}(G)|}$, where $\text{Aut}(G)$ is understood as the automorphism group of $G$ with any labeling. Thus

$$\sum_{\text{$G$ unlabeled graph with $n$ vertices}} \frac{n!}{|\text{Aut}(G)|} = 2\binom{n}{2}.$$

- The number of unlabeled graphs with $n$ vertices is at least $\lceil \frac{2\binom{n}{2}}{n!} \rceil$.

**Intersection Graphs, Interval Graphs:**

- Let $\mathcal{F}$ be a family of subsets of a set $V$. The **intersection graph** of $\mathcal{F}$ is a graph whose vertex set is $\mathcal{F}$, and two subsets of $\mathcal{F}$ are adjacent if their intersection is nonempty.

- Let $V = \mathbb{R}$ and $\mathcal{F}$ be a set of closed intervals of $\mathbb{R}$. The intersection graph of $\mathcal{F}$ is called an **interval graph**.

**Cayley Graphs**

- Let $\Gamma$ be a group. Given a subset $S$ of $\Gamma$ such that $S$ does not contain the identity element and is closed under inverse operation. The **Cayley graph** of $\Gamma$ with respect to $S$ is a graph $G(\Gamma, S)$ with the vertex set $\Gamma$ in which two vertices $x, y$ are adjacent if $xy^{-1} \in S$.

**Polyhedral Graphs:**

**Infinite Graphs:**

## 2 Subgraphs

**Definition of Subgraphs:**

- A graph $H$ is called a **subgraph** of a graph $G$ if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and $\text{End}_H : E(H) \to M_2(V(H))$ is the restriction of $\text{End}_G : E(G) \to M_2(V(G))$ to $E(H)$. We then say that $G$ **contains** $H$ or $H$ is **contained in** $G$, and we write $H \subseteq G$ or $G \supseteq H$.

- A **copy** of a graph $H$ in a graph $G$ is a subgraph of $G$ which is isomorphic to $H$. Such a subgraph is also referred to as an **$H$-subgraph** of $G$.

- An **embedding** of a graph $H$ in a graph $G$ is an isomorphism from $H$ to a subgraph of $G$. For each copy of $H$ in $G$, there are $|\text{Aut}(H)|$ embeddings in $G$.

- A maximal connected subgraph of a graph $G$ is called a **connected component** of $G$. The number of connected components of $G$ is denoted by $c(G)$.

**Deletion, Contraction:**

- Let $v$ be a vertex in a graph $G$. We denote by $G - v$ the graph obtained from $G$ by deleting the vertex $v$ and all the edges incident with $v$. Such an operation is referred to as a **vertex deletion**, and $G - v$ as a **vertex-deleted subgraph**.
Theorem 2.1. A graph $G$ whose every vertex has degree at least 2 contains a cycle.

Proof. Let $P := v_0v_1 \cdots v_{\ell-1}v_\ell$ be a longest path in $G$; such a path do exists since $G$ is finite. If $v_0 = v_\ell$, then $P$ is already a cycle. Otherwise, the degree of $v_0$ in $P$ is 1. Since the degree of $v_0$ in $G$ is at least 2, there is an edge $e_0$ (not in $P$) joining $v_0$ to a vertex $v$. If $v = v_i$ for some $0 \leq i \leq \ell - 1$, then $P_i = v_0v_1 \cdots v_i$ is a cycle. Otherwise $Q := v_0P$ is a path longer than $P$, a contradiction. \qed

Corollary 2.2. A graph without cycles has at least one vertex of degree 0 or 1.

Acyclic Graphs:

- A graph is said to be **acyclic** if it does not contain any cycle. Acyclic graphs are usually called **forests**. A connected acyclic graph is usually called a **tree**.
- A vertex of degree 1 in a tree is called a **leaf** of the tree.
- A tree with at least one edge has at least two leaves.

Spanning Subgraphs, Induced Subgraphs:

- A **spanning subgraph** $H$ of a graph $G$ is a subgraph such that $V(H) = V(G)$.
- The **symmetric difference** of spanning subgraphs $G_1$ and $G_2$ of a graph $G = (V, E)$ is a spanning subgraph of $G$ whose edge set is $E(G_1) \Delta E(G_2)$.
- Let $X$ be a vertex subset of a graph $G$. An **induced subgraph** by $X$ is a graph $G[X]$ whose vertex set is $X$ and whose edge set consists of all edges of $G$ which have end-vertices in $X$.
- Let $S$ be an edge subset of a graph $G$. An **induced subgraph** by $S$ is a graph $G[S]$ whose edge set is $S$ and whose vertex set consists of all end-vertices of edge in $S$.

Decompositions, Coverings:

- A **decomposition** of a graph $G$ is a family of edge-disjoint subgraphs of $G$ such that
  \[ E(G) = \bigcup_{H \in \mathcal{F}} E(H). \]
  A decomposition $\mathcal{F}$ is referred to a **path (cycle) decomposition** if the family $\mathcal{F}$ consists entirely of path (cycles) of $G$.
- A graph is said to be **even** if every vertex has even degree.
- A **covering** or **cover** of a graph $G$ is a family $\mathcal{F}$ of not necessarily edge-disjoint subgraphs of $G$ such that
  \[ E(G) = \bigcup_{H \in \mathcal{F}} E(H). \]
  A covering $\mathcal{F}$ is referred to a **path (cycle) covering** if the family $\mathcal{F}$ consists entirely of path (cycles) of $G$.
- A covering of a graph $G$ is said to be **uniform** if every edge of $G$ is covered the same number of times by $\mathcal{F}$. When this number is $k$, the covering is called a **$k$-cover**. A 2-cover is usually called a **double cover**.
Theorem 2.3. A graph admits a cycle decomposition if and only it is even.

Proof. The necessity is trivial, for the degree of every vertex of a cycle is 2 and the degree of a vertex in the graph is a summation of 2’s.

Let G be an even graph. If G contains some edges, then G contains a cycle C1 by Theorem 2.1. Remove the edges of C1 from G to obtain a graph G1, which is still even. Then by Theorem 2.1 again there is a cycle C2 in G1. Remove the edges of C2 from G1 to obtain a graph G2, which is even. Continue this procedure; we obtain a cycle decomposition of G.

Theorem 2.4. Let \( \mathcal{F} = \{ F_1, F_1, \ldots, F_k \} \) be a decomposition of \( K_n \) of bipartite graphs. Then \( k \geq n - 1 \).

Cuts, Bonds

- Let X and Y be vertex subsets of a graph G or digraph D. We denote by \([X,Y]\) the set of edges with one end-vertex in X and the other end-vertex in Y, and by \((X,Y)\) to the set of directed edges with the tail in X and the head in Y. An edge set of the form \([X,X^c]\) is called an edge cut or cut, where \(X^c\) is the complement of X in \(V(G)\).
- For any vertex subset of a graph G,
  \[ ||X,X^c|| + 2||X,X|| = \sum_{v \in X} d_G(v). \]
- A bond of a graph G is a minimal nonempty cut, i.e., a nonempty edge cut none of whose nonempty proper subset is an edge cut.
- Deleting the edges of a cut increases the number of connected components.

Theorem 2.5. A graph G is even if and only if every cut of G has even number of edges.

Proof. If G is even, then for any subset \( X \subseteq V(G) \), \( ||X,X^c|| = -2||X,X|| + \sum_{v \in X} d_G(v) \) is clearly even. Conversely, we have a cut \([\{v\}, \{v\}^c]\) for each vertex \( v \in V(G) \). The degree \( d_G(v) = ||\{v\}, \{v\}^c|| + 2||\{v\}, \{v\}|| \) is clearly even.

Proposition 2.6. Let X and Y be vertex subsets of a graph G. Then

\[ [X,X^c] \Delta [Y,Y^c] = [X \Delta Y, (X \Delta Y)^c]. \]

Proof. Note that \( \{X \cap Y, X \cap Y^c, X^c \cap Y, X^c \cap Y^c\} \) is a partition of \( V \).

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Since \([X,X^c] \cap [Y,Y^c] = [X \cap Y, X^c \cap Y^c]\), we have

\[ [X,X^c] \Delta [Y,Y^c] = [X,X^c] \cup [Y,Y^c] - [X \cap Y, X^c \cap Y^c] = [X \cap Y, X \cap Y^c] \cup [X \cap Y, X^c \cap Y] \cup [X \cap Y^c, X^c \cap Y^c] \cup [X^c \cap Y, X^c \cap Y^c]. \]

Since \( X \Delta Y = (X \cap Y^c) \cup (X^c \cap Y) \) and \( (X \Delta Y)^c = (X \cap Y) \cup (X^c \cap Y^c) \), we have

\[ [X \Delta Y, (X \Delta Y)^c] = [(X \cap Y^c) \cup (X^c \cap Y), (X \cap Y) \cup (X^c \cap Y^c)] = [X \cap Y^c, X \cap Y] \cup [X \cap Y^c, X^c \cap Y] \cup [X \cap Y^c, X^c \cap Y^c] \cup [X \cap Y^c, X^c \cap Y^c]. \]

Note that \([X \cap Y, X \cap Y^c] = [X \cap Y^c, X \cap Y] \) and \([X \cap Y, X^c \cap Y] = [X^c \cap Y, X \cap Y]\).
Proposition 2.7. Let \( B \) be an edge subset of a connected graph \( G \). Then \( B \) is a bond if and only if there is a vertex subset \( X \) such that both \( G[X] \) and \( G[X^c] \) are connected and \( B = [X, X^c] \).

Proof. “\( \Rightarrow \)” Since \( B \) is a cut, i.e., there is a vertex subset \( X \) such that \( B = [X, X^c] \). We claim that both \( G[X] \) and \( G[X^c] \) are connected. Suppose \( G[X] \) is disconnected, say \( X = X_1 \cup X_2, X_i \neq \emptyset, i = 1, 2 \), and there are no edges between \( X_1 \) and \( X_2 \). Then both \([X_1, X^c]\) and \([X_2, X^c]\) are nonempty cuts, and are contained in \([X, X^c]\); this contradicts to that \( B \) is a minimal cut. So \( G[X] \) is connected; and similarly for \( G[X^c] \).

“\( \Leftarrow \)” Clearly, \( B = [X, X^c] \) is a cut. Suppose \( B \) is not minimal, i.e., there is a proper subset \( B_1 \subseteq B \) such that \( B_1 \) is also a cut. Then \( G - B_1 \) is disconnected. However, there exists an edge \( e \in B - B_1 \), \( G[X] \) and \( G[X^c] \) are connected. Then \( G - B_1 \) is connected, a contradiction. \( \square \)

Proposition 2.8. An edge subset of a graph \( G \) is a cut if and only if it is a disjoint union of bonds.

Proof. The sufficiency is trivial. For necessity, consider an edge cut \([X, X^c]\) of \( G \). Let \( G[X] \) be decomposed into connected components and let \( G_1, \ldots, G_k \) be those components having a vertex adjacent to a vertex in \( X \). Set \( X_i := V(G_i), 1 \leq i \leq k \). Then \([X, X^c]\) is a disjoint union of the edge cuts \([X_i, X_i^c]\). Fix an index \( i \), let \( G[X_i] \) be decomposed into connected components and let \( H_1, \ldots, H_l \) be those components having a vertex adjacent to a vertex in \( X_i \). Set \( Y_j := V(H_j) \). Then \([X_i, X_i^c]\) is a disjoint union of the edge cuts \([Y_j, Y_j^c]\). We claim that each \([Y_j, Y_j^c]\) is a bond. In fact, \([Y_j, Y_j^c]\) consists of all the edges between the connected subgraphs \( G_i \) and \( H_j \). Suppose there is a proper subset \( S \) of \([X_i, X_i^c]\) such that \( S = [Z, Z^c]\) is an edge cut. Since there are edges (other than the edges of \( S \)) joining \( G_i \) and \( H_j \), then both \( G_i \) and \( H_j \) must be subgraphs of either \( G[Z] \) or \( G[Z^c] \). However, \( S \) is a set of some edges between \( G_i \) and \( H_j \), so \( S \) is an edge subset of either \( G[Z] \) or \( G[Z^c] \). This is a contradiction. \( \square \)

Vector Spaces Associated with Graphs:

- Let \( S \) be a set and \( \mathbb{F} \) a field. Let \( \mathbb{F}^S \) be the set of all functions from \( S \) to \( \mathbb{F} \). The \( \mathbb{F}^S \) becomes a vector space over \( \mathbb{F} \) under the following addition and scalar multiplication: For \( f, g \in \mathbb{F}^S \) and \( c \in \mathbb{F} \),
  
  \[(f + g)(s) = f(s) + g(s), \quad (cf)(s) = cf(s), \quad s \in S.\]

- Let \( S \) be a set and \( \mathbb{F}_2 = \{0, 1\} \) a field of two elements. There is a one-to-one correspondence between the power set \( \mathcal{P}(S) \) and the vector space \( \mathbb{F}_2^S \). In fact, a subset \( A \subseteq S \) corresponds to its characteristic function \( 1_A : S \rightarrow \mathbb{F}_2 \), written \( A \leftrightarrow 1_A \), where
  
  \[1_A(s) = \begin{cases} 1 & \text{if } s \in A, \\ 0 & \text{if } s \notin A. \end{cases} \]

  Moreover, for subsets \( A, B \subseteq S \), we have
  
  \[A \Delta B := A \cup B - A \cap B \leftrightarrow 1_A + 1_B.\]

  So \( \mathcal{P}(S) \) can be considered as a vector space of dimension \( |S| \), where the zero vector is the empty set.

- For a graph \( G = (V, E) \), The vector space \( \mathbb{F}_2^V \) is called the vertex space and \( \mathbb{F}_2^E \) the edge space of \( G \).

- The set of even graphs of a graph \( G \) forms a subspace of the edge space of \( G \), called the cycle space of \( G \).

- The set of all cuts of a graph \( G \) forms a subspace of the edge space of \( G \), called the bond space of \( G \).

Proposition 2.9. Let \( G_i = (V, E_i) \) be spanning subgraphs of a graph \( G = (V, E) \). Then for any \( X \subseteq V(G) \),

\[\partial_{G_1 \Delta G_2}(X) = \partial_{G_1}(X) \Delta \partial_{G_2}(X).\]

Proof.

\[
\begin{align*}
\partial_{G_1 \Delta G_2}(X) &= [X, X^c] \cap (E_1 \Delta E_2) \\
&= [X, X^c] \cap (E_1 \cup E_2 - E_1 \cap E_2) \\
&= (([X, X^c] \cap E_1) \cup ([X, X^c] \cap E_2)) - [X, X^c] \cap E_1 \cap E_2 \\
&= ([X, X^c] \cap E_1) \Delta ([X, X^c] \cap E_2) \\
&= \partial_{G_1}(X) \Delta \partial_{G_2}(X).
\end{align*}
\]

\( \square \)
Proof. Suppose it is not true, i.e., the maximal degree \( \Delta(G) < n - 1 \). We first show that \( G \) is regular. Consider two non-adjacent vertices \( x \) and \( y \). We define a function \( f : N(x) \to N(y) \), where for each \( v \in N(x) \), \( f(v) \) is defined as the unique common neighbor of \( v \) and \( y \). Then \( f \) is injective. In fact, if \( f(u) = f(v) \) for distinct \( u, v \in N(x) \), then \( u \) and \( v \) have two common neighbors \( x \) and \( f(u) \), a contradiction. Thus \( d(x) = |N(x)| \leq |N(y)| = d(y) \). Similarly, \( d(y) \leq d(x) \). So \( d(x) = d(y) \). This is equivalent to say that any two adjacent vertices of \( \bar{G} \) (the complement simple graph of \( G \)) have the same degree. We claim that \( G \) is regular.

To this end, it suffices to show that \( \bar{G} \) is connected. Note that \( G \) has no single vertices, since the minimal degree \( \delta(G) = n - 1 - \Delta(G) > 0 \). Suppose \( \bar{G} \) has two or more connected components. Take two edges \( e_i = u_iv_i \) from distinct components of \( \bar{G} \), \( i = 1, 2 \). Then \( u_1u_2v_1v_2u_1 \) is a cycle of \( G \). Thus \( u_1 \) and \( v_1 \) have at least two common neighbors \( u_2 \) and \( v_2 \), a contradiction. Let \( G \) be \( k \)-regular. Consider the number of paths of length 2 in \( G \). Since any two vertices have a unique common neighbor, there \( \binom{n}{2} \) paths of length 2. For each vertex \( v \), there are \( \binom{k}{2} \) paths with the middle vertex \( v \). Hence \( \binom{n}{2} = n \binom{k}{2} \), i.e., \( n = k^2 - k + 1 \).

Let \( A \) be the adjacency matrix of \( G \). The \( (u, v) \)-entry of \( A^2 \) is the number of \( (u, v) \)-walks of length 2. Then \( A^2 \) has its diagonal entries \( k \) and other entries 1. So \( A^2 = (k - 1)I + J \), where \( I \) is the identity matrix and \( J \) is a matrix whose entries are 1. Note that \( J \) has the eigenvalue 0 with multiplicity \( n - 1 \) and the eigenvalue \( n \). Then \( A^2 \) has the eigenvalue \( k - 1 \) with multiplicity \( n - 1 \) and the eigenvalue \( k^2 (= n + k - 1) \) with the multiplicity 1. It follows that \( A \) has the eigenvalue \( \pm \sqrt{k - 1} \) with multiplicity \( n - 1 \) and the eigenvalue \( k \) with the multiplicity 1.

Since the graph \( G \) is simple, the trace of \( A \) (the sum of diagonal entries) is zero. Note that the trace of \( A \) is also the sum of its eigenvalues (counted with multiplicities). Then \( \pm (n - 1) \sqrt{k - 1} + k = 0 \); it forces that \( (n - 1) \sqrt{k - 1} + k = 0 \). The only possible choice is that \( k = 2 \) and \( n = 3 \), i.e., \( G \) is a triangle, where \( \Delta(G) = 2 \). This is contradict to that \( \Delta(G) < n - 1 \). \( \square \)