Problem 1. Consider an \(m\)-by-\(n\) chessboard in which certain squares are forbidden. What is the largest number of non-attacking rooks that can be placed on the board?

Problem 2. Consider again an \(m\)-by-\(n\) chessboard where certain squares are forbidden. What is the largest number of dominoes that can be placed on the board so that each domino covers two allowed squares and no two dominoes overlap?

Problem 3. A company has \(n\) jobs available, with each job demanding certain qualifications. There are \(m\) people who apply for the \(n\) jobs. What is the largest number of jobs that can be filled from the available \(m\) applicants if a job can be filled only by a person who meets its qualifications?

1 General Problem Formulation

Let \(G = (X, \Delta, Y)\) be a bipartite graph. A **matching** of \(G\) is a subset \(M\) of the set \(\Delta\) of edges, with the property that no two of the edges of \(M\) have a common vertex.

**Example 1.1.** Let \(X = \{x_1, x_2, x_3, x_4\}\) and \(Y = \{y_1, y_2, y_3, y_4, y_5\}\). Consider the 4-by-5 board, whose rows are indexed by elements of \(X\) and whose columns
are indexed by elements of $Y$, and with forbidden positions shown below:

<table>
<thead>
<tr>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$y_3$</th>
<th>$y_4$</th>
<th>$y_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td></td>
<td></td>
<td></td>
<td>$\times$</td>
</tr>
<tr>
<td>$x_2$</td>
<td></td>
<td></td>
<td>$\times$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$x_3$</td>
<td></td>
<td>$\times$</td>
<td>$\times$</td>
<td></td>
</tr>
<tr>
<td>$x_4$</td>
<td>$\times$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We associate with this board a bipartite graph $G = (X, \Delta, Y)$ whose vertex set is $X \cup Y$ and the edge set $\Delta$ is defined as follows:

\[
\{x_i, y_j\} \in \Delta \iff \text{the square in the } i\text{th row and } j\text{th column is allowed.}
\]

The graph $G = (X, \Delta, Y)$ is called a rook-bipartite graph. In this example, the edge set $\Delta$ is given by

\[
\Delta = X \times Y \setminus \{\{x_1, y_2\}, \{x_2, y_4\}, \{x_2, y_5\}, \{x_3, y_3\}, \{x_3, y_5\}, \{x_4, y_1\}\}.
\]

**Example 1.2.** Consider a 4-by-5 board with some forbidden squares and whose non-forbidden squares are alternately colored black and white. For identification we label the non-forbidden black squares $b_1, b_2, \ldots, b_6$ and white squares $w_1, w_2, \ldots, w_7$; see below.

<table>
<thead>
<tr>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$b_1$</th>
<th>$w_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_2$</td>
<td>$w_4$</td>
<td>$\times$</td>
<td>$\times$</td>
</tr>
<tr>
<td></td>
<td>$b_3$</td>
<td>$w_5$</td>
<td>$b_4$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$w_6$</td>
<td>$b_5$</td>
</tr>
</tbody>
</table>

We define a bipartite graph $G = (X, \Delta, Y)$, where $X = \{b_1, b_2, \ldots, b_6\}$, $Y = \{w_1, w_2, \ldots, w_7\}$, and

\[
\{b_i, w_j\} \in \Delta \iff \text{the squares } b_i \text{ and } w_j \text{ have a common side.}
\]

**Example 1.3.** Four people $x_1, x_2, x_3, x_4$ apply for five jobs $y_1, y_2, y_3, y_4, y_5$. Suppose that (1) the person $x_1$ is qualified for the jobs $y_1, y_2, y_4, y_5$; (2) the person $x_2$ is qualified for the jobs $y_2, y_3, y_4$; (3) the person $x_3$ is qualified for the jobs $y_2, y_5$; and (4) the person $x_4$ is qualified for the jobs $y_1, y_2, y_4, y_5$. We can construct a bipartite graph $G = (X, \Delta, Y)$, where $X = \{x_1, x_2, x_3, x_4\}$, $Y = \{y_1, y_2, y_3, y_4, y_5\}$, and

\[
\{x_i, y_j\} \in \Delta \iff \text{the person } x_i \text{ is qualified for the job } y_j.
\]
2 Matchings

Definition 2.1. A subset $M$ of $E$ is called a **matching** of a graph $G = (V, E)$ if no two edges of $M$ share a common vertex; if so the two endpoints of an edge in $M$ are said to be **matched under** $M$. A matching $M$ is called **maximum** if there is no matching $M'$ such that $|M'| > |M|$. The **matching number** of $G$ is defined as

$$m(G) = \max \{ |M| : M \text{ is a matching of } G \}.$$  

A matching $M$ is called **perfect** if every vertex of $G$ is matched under $M$.

Definition 2.2. Let $M$ be a matching of a graph $G$. A vertex $v$ of $G$ is called **saturated** by $M$ (or $M$-saturated) if $v$ is incident with an edge of $M$; otherwise it is called **unsaturated** by $M$ (or $M$-unsaturated). A path $P$ of $G$ is called **$M$-alternating** if the edges of $P$ alternate between $M$ and its complement $\overline{M} := E \setminus M$. An $M$-alternating path is called **$M$-augmenting** if its initial and terminal vertices are not matched under $M$.

For any $M$-augmenting path $P$ of a graph $G$, the length of $P$ must be odd; the 1st, 3rd, 5th, ..., edges of $P$ are contained in the complement $\overline{M}$, that is, the odd edges of $P$ belong to $\overline{M}$; and the 2nd, 4th, 6th, ..., edges of $P$ are contained in $M$, that is, the even edges of $P$ belong to $M$.

Proposition 2.3. Let $M$ be a matching of a graph $G$, and let $P$ be an $M$-alternating path in $G$. Then

(a) $P$ has no self-intersect vertices;

(b) $M' := (M \setminus P) \cup (P \setminus M)$ is a matching of $G$ and $|M'| = |M| + 1$.

Proof. (a) Suppose the path $P = v_0v_1 \ldots v_{2m+1}$ has self-intersection, that is, two of the vertices $v_0, v_1, \ldots, v_{2m+1}$ are the same, say $v_i = v_j$ with $i < j$. There are three possibilities: (1) $0 < i < j < 2m = 1$; (2) $0 < i < j = 2m + 1$; and (3) $0 = i < j < 2m + 1$.

In the first case there are must two edges of $M$ incident with the intersection vertex $v_2 = v_8$. This is impossible because $M$ is a matching. In the second and third cases, the intersection vertex is either the beginning vertex $v_0$ or the ending
vertex \( v_{2m+1} \), and there is at least one edge of \( M \) incident with the intersection vertex. This is also impossible because both \( v_0 \) and \( v_{2m+1} \) are not incident with any edge of \( M \).

(b) Since \( M \) is a matching, then \( M - P \) has no two edges sharing a common vertex. Since \( P \) has no self-intersection, its obvious that \( P - M \) has no two edges sharing a vertex. Note that \( M - P \) and \( P - M \) have disjoint vertex sets. It follows that \( M' = (M - P) \cup (P - M) \) is a matching. Obviously, \(|M'| = |M| + 1\).

\[ \square \]

**Theorem 2.4** (Berge, 1957). *Let \( M \) be a matching of a graph \( G \). If there is no \( M \)-augmenting path, then \( M \) is a maximal matching of \( G \).*

**Proof.** Suppose \( M \) is a not maximal matching. Then there is a matching \( M' \) such that \(|M'| > |M|\). Consider the graph \( G^* = (V, E^*) \), where \( V \) is the same vertex of \( G \) and \( E^* \) is the the symmetric sum of \( M \) and \( M' \), that is,

\[ E^* = (M - M') \cup (M' - M). \]

Since \(|M'| > |M|\), we have

\[ |M' - M| > |M - M'|. \]

The graph \( G^* \) has the property that each vertex is incident with at most two edges of \( \Delta^* \), at most one edge in \( M - M' \) and at most one edge in \( M' - M \). This means that the degree of every vertex is at most two. Thus the edge set \( \Delta^* \) can be partitioned into paths (without self-intersections) and cycles. The paths and cycles must be \( M \)-alternating and can be classified into four types:

**Type 1.** A path whose first and last edges are both in \( M' - M \).

**Type 2.** A path whose first and last edges are both in \( M - M' \).

**Type 3.** A path whose first edge is in \( M - M' \) and whose last edges is in \( M' - M \), or the first edge is in \( M' - M \) and the last edge is in \( M - M' \).
Type 4. A cycle.

Note that a Type 1 path has more edges in $M'$ than the edges in $M$; a Type 2 path has more edges in $M$ than the edges in $M'$; a Type 3 path has equal number of edges in both $M$ and $M'$; and a Type 4 cycle has the same number of edges in both $M$ and $M'$. Since $|M' - M| > |M - M'|$, there must exist a Type 1 path $P = v_0v_1 \ldots v_{2m+1}$. If the beginning vertex $v_0$ is incident with an edge $e$ in $M$, then $e$ belongs to $M - M'$; thus $v_0$ can not be the beginning vertex, a contradiction. So $v_0$ is not incident with any edge of $M$. Similarly, the ending vertex $v_{2m+1}$ is not incident with any edge of $M$. Thus $P$ is an $M$-augmenting path. This is a contradiction. □

3 Coverings

Definition 3.1. A subset $C$ of $V$ in a graph $G = (V, E)$ is called a covering of $G$ if every edge of $G$ is incident with a vertex in $C$.

A covering $C$ is said to be minimum if there is no covering $C'$ such that $|C'| < |C|$. The covering number of $G$ is defined as

$$c(G) = \min \{|C| : C \text{ is a covering of } G\}.$$ 

Let $M$ be a matching and $C$ a covering of a graph $G$. Since each edge of $M$ is covered by a vertex in $K$ and distinct edges of $M$ must be covered by distinct vertices of $C$, so $|M| \leq |C|$. We then have

$$m(G) \leq c(G).$$

It may be speculated that $m(G) = c(G)$. Unfortunately, this is not true in general. However, if $G$ is bipartite, the equality $m(G) = c(G)$ does hold.

Lemma 3.2. Let $M$ be a matching and let $C$ a covering of a graph $G$. If $|M| = |C|$, then $M$ is a maximum matching and $C$ is a minimum covering.

Theorem 3.3 (König, 1931). If $G$ is a bipartite graph, then $m(G) = c(G)$.

In words it states that the number of edges in a maximum matching is equal to the number of vertices in a minimum covering.
Proof. It is known that \( m(G) \leq c(G) \). We only need to show that \( m(G) \geq c(G) \).

Let \( M \) be a matching and \( C \) a covering of \( G \).

Let \( G \) be a bipartite graph with the bipartition \((X,Y)\), and let \( M^* \) be a maximum matching of \( G \). Denote by \( U \) the set of all \( \square \)

**Algorithm 3.4** (Labelling Algorithm). Let \( G = (X, \Delta, Y) \) be a bipartite graph, where

\[
X = \{x_1, x_2, \ldots, x_m\} \quad \text{and} \quad Y = \{y_1, y_2, \ldots, y_n\}.
\]

Let \( M \) be any matching of \( G \). All vertices are **unscanned**.

**Step 0.** Label with the symbol (\( \ast \)) all vertices of \( X \) unsaturated by \( M \). If there is no \( M \)-unsaturated vertex in \( X \), then Stop; \( M \) is already a maximum matching and \( X \) is a minimum covering. Otherwise, go to Step 1.

**Step 1.** Select a labeled but unscanned vertex \( x \) of \( X \); label with (\( x \)) all unlabeled vertices of \( Y \) that are adjacent to \( x \) by an edge not in \( M \). The vertex \( x \) is called **scanned**. Repeat this until all labeled but unscanned vertices of \( X \) are scanned. Now, if no new label has been given to a vertex of \( Y \), then Stop; \( M \) is a maximum matching and \( S = X^{un} \cup Y^{lab} \) is a minimum covering, where \( X^{un} \) consists of the unlabeled vertices of \( X \) and \( Y^{lab} \) consists of the labeled vertices of \( Y \). Otherwise, go to Step 2.
Step 2. If there exists a labeled but unscanned vertex \( y \) that is \( M \)-unsaturated, then Stop; chasing back from \( y \) through labels in reversing order, an \( M \)-augmenting path 

\[
P(y) = yx_1y_1x_2y_2 \cdots x_my_mx_{m+1}
\]

can be constructed as follows: the label of \( y \) is \( x_1 \), the label \( x_i \) is \( (y_i) \) and the label of \( y_i \) is \( (x_{i+1}) \) for all \( 1 \leq i \leq m \), and the label of \( x_{m+1} \) is \( (*) \).

If all labeled but unscanned vertices of \( Y \) are \( M \)-saturated, select such a vertex \( y \) and label with \( (y) \) the vertex of \( X \) that is adjacent to \( y \) by an edge in \( M \). The vertex \( y \) is called \textbf{scanned}. Repeat this until all labeled but unscanned vertices of \( Y \) are scanned. Then go to Step 1.

**Proof.** Since each vertex receives at most one label and is scanned at most once, the Labelling Algorithm halts after a finite number of steps.

**Case 1.** Algorithm 3.4 stops in Step 1.

We first show that the set \( S = X^{un} \cup Y^{lab} \) is a covering. Suppose that this is not true, that is, there is an edge \( e = (x, y) \), where \( x \in X \) and \( y \in Y \), such that \( x \notin S \) and \( y \notin S \). Then \( x \) is labeled and \( y \) is unlabeled. The edge \( e \) is either in \( M \) or not in \( M \). If \( e \) is not in \( M \), then by Step 1 the vertex \( y \) receives the label \( (x) \); this is a contradiction. If \( e \) is in \( M \), then \( x \) is \( M \)-saturated. Thus the label of \( x \) can not be \( (*) \) by Step 0; the label of \( x \) must be \( (y) \) by Step 2. This means that the vertex \( y \) is already labeled, contradictory to that \( y \) is unlabeled.

Next we claim that \( |M| = |S| \). This equality implies that \( M \) is a maximum matching and \( S \) is a minimum covering. To show this equality we establish a bijection from \( S \) to \( M \). For each vertex \( y \in Y^{lab} \), the vertex \( y \) must be \( M \)-saturated since Algorithm 3.4 stops in Step 1. Then there is one and only one edge of \( M \) incident with \( y \), say \( e = (x, y) \). Hence \( x \) receives the label \( (y) \) in Step 2, so \( x \notin X^{un} \). Thus each vertex of \( Y^{lab} \) incident with one edge of \( M \) whose other endpoint belongs to \( X - X^{un} \). Now for each vertex \( x' \in X^{un} \), since \( x' \) is not labeled, the vertex \( x' \) must be \( M \)-saturated (otherwise \( x \) receives the label \( (*) \)). Then there is one and only one edge of \( M \) incident with \( x' \), say \( e' = (x', y') \).
This edge $e'$ is distinct from any edge $e = (x, y)$ of $M$ incident with a vertex $y \in Y^{lab}$. Thus
\[ |S| = |X^{un} \cup Y^{lab}| \leq |M|. \]
Since $|M| \leq |S|$, we conclude that $|M| = |S|$.

**Case 2.** Algorithm 3.4 stops in Step 2.

Let $(y_1)$ be the label of $y$. Then $x_1$ must be labeled before $y$ in Step 2. Let $(y_1)$ be the label of $x_1$, then $y_1$ must be labeled before $x_1$ in Step 1. Let $(x_2)$ be the label of $y_1$, $(y_2)$ be the label of $x_2$, ..., $(y_m)$ be the label of $x_m$, $(x_{m+1})$ be the label of $y_m$, and finally let $(\ast)$ be the label of $x_{m+1}$. Then we have a path $P = y_0x_1y_1x_2y_2 \cdots x_my_mx_{m+1}$. It follows from Step 1 and Step 2 that the edges
\[ (y, x_1) \in \bar{M}, \quad (x_i, y_i) \in M, \quad (y_i, x_{i+1}) \in \bar{M} \quad (1 \leq i \leq m). \]
This means that $P$ is an $M$-altering path. Since the beginning and ending vertices $y, x_{m+1}$ are $M$-unsaturated, the path $P$ is an $M$-augmenting path. □

The Algorithm 3.4 can be easily modified to give an algorithm to find a maximum matching and a minimum covering for a bipartite graph. We give such an algorithm in the following.

**Algorithm 3.5** (Matching-Covering Algorithm). Let $G = (X, \Delta, Y)$ be a bipartite graph, where
\[ X = \{x_1, x_2, \ldots, x_m\} \quad \text{and} \quad Y = \{y_1, y_2, \ldots, y_n\}. \]

Step 0. Begin with a matching $M$ and consider all vertices un-scanned. Label with the symbol $(\ast)$ all vertices of $X$ unsaturated by $M$. If there is no $M$-unsaturated vertex in $X$, then Stop; $M$ is a maximum matching, and $X$ is a minimum covering. Otherwise, go to Step 1.

Step 1. If there is no labeled but unscanned vertex of $X$, then go to Step 2. Otherwise, select a labeled but unscanned vertex $x$ of $X$; label with $(x)$ all unlabeled vertices of $Y$ that are adjacent to $x$ by an edge not in $M$. The vertex $x$ is called scanned. Then return to Step 1.
Step 2. If no new label has been given to a vertex of $Y$, then Stop; $M$ is a maximum matching and $S = X^{un} \cup Y^{lab}$ is a minimum covering. Otherwise, go to Step 3.

Step 3. If there is no unscanned labeled vertex of $Y$, then go to Step 1. Otherwise, go to Step 4.

Step 4. Select a labeled but unscanned vertex $y$ of $Y$. If $y$ is $M$-unsaturated, then chase back from the labels to construct an $M$-augmenting path $P(y)$, set $M := M \Delta P(y)$, and go to Step 0. If $y$ is $M$-saturated, then label with $(y)$ the vertex of $X$ that is adjacent to $y$ by an edge in $M$. The vertex $y$ is called scanned. Then go to Step 3.

4 Systems of Distinct Representatives

Let $\mathcal{A} = (A_1, A_2, \ldots, A_n)$ be a family of subsets of a set $X$. A family $S = (a_1, a_2, \ldots, a_n)$ of elements of $X$ is called a system of representatives for $\mathcal{A}$ if

$$a_1 \in A_1, \ a_2 \in A_2, \ \ldots, \ a_n \in A_n;$$

and is further called a system of distinct representatives (SDR) if, in addition to that $S$ is a system of representatives, all the elements $a_1, a_2, \ldots, a_n$ are distinct.

Example 4.1. Let $(A_1, A_2, A_3, A_4)$ be a family of subsets of the set

$$Y = \{a, b, c, d, e\}$$

defined by

$$A_1 = \{a, b, c\}, \ A_2 = \{b, d\}, \ A_3 = \{a, b, d, e\}, \ A_4 = \{a, d, e\}.$$ 

Then $(c, d, d, e)$ is a system of representatives, and $(b, d, e, a)$ is an SDR.

Let $\mathcal{A} = (A_1, A_2, \ldots, A_n)$ be a family of sets. If $\mathcal{A}$ has an SDR

$$S = (a_1, a_2, \ldots, a_n),$$
then for any selection \( \{i_1, i_2, \ldots, i_k\} \) of indices, the union
\[
A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k}
\]
contains the subset \( \{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\} \). Thus we have
\[
(MC) : \quad |A_{i_1} \cup \cdots \cup A_{i_k}| \geq k, \quad \forall i_1 < \cdots < i_k.
\]
The condition (1) is known as the **Marriage Condition (MC)** because of the following Marriage Problem. The condition (1) is also known as **Hall’s Condition** because of Theorem 4.1.

**Example 4.2** (Marriage Problem).

**Theorem 4.1** (Hall, 1935). A family \( \mathcal{A} = (A_1, A_2, \ldots, A_n) \) of sets has an SDR if and only if the Marriage Condition (MC) holds.

*Proof.* We have seen that MC is necessary. We only need to prove that MC is sufficient. Let \( G \) be the bipartite graph whose vertex set is the union \( X \cup Y \), where
\[
X = \{1, 2, \ldots, n\}, \quad Y = A_1 \cup A_2 \cup \cdots \cup A_n,
\]
and whose edge set is
\[
E = \{(x, y) \in X \times Y : y \in A_x\}.
\]
To show that \( \mathcal{A} \) has an SDR, it is equivalent to show that the bipartite graph \( G \) has the matching number \( m(G) = n \). It then suffices to show that the covering number \( c(G) = n \).

Suppose this is not true, that is, there is a covering \( C \) such that \( |C| < n \). Let \( C_1 = C \cap X \) and \( C_2 = C \cap Y \). Then
\[
|C_1| + |C_2| < n.
\]
Since \( |C_1| < n \) and \( |X| = n \), then \( X \setminus C_1 \) is nonempty. We write
\[
X \setminus C_1 = \{i_1, i_2, \ldots, i_k\}.
\]
Since \( C \) is a covering, there is no edge of \( G \) from a vertex in \( X \setminus C_1 \) to a vertex in \( Y \setminus C_2 \). This means that all the sets \( A_{i_1}, A_{i_2}, \ldots, A_{i_k} \) must be contained in \( C_2 \). Hence \( A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k} \) is a subset of \( C_2 \). Thus
\[
|A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k}| \leq |C_2|.
\]
Since $|C_1| + |C_2| < n$ and $|C_1| = n - k$, we have 

$$|C_2| < n - |C_1| = k.$$ 

Therefore 

$$|A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k}| < k,$$

which is contradictory to the Marriage Condition. \hfill $\Box$

**Direct Proof by Induction:** For $n = 1$, it is trivially true.

Assume $n \geq 2$. There are two cases to be considered: the **tight case** and the **room to spare case**.

**The Tight Case:** There is an inter $k$ with $1 \leq k \leq n - 1$ and subfamily of $\mathcal{A}$ of $k$ sets whose union contains exactly $k$ elements. (By MC the union cannot contain fewer than $k$ elements, so we are tight.) Without loss of generality, we may assume that the $k$ sets are the first $k$ sets $A_1, \ldots, A_k$. Set 

$$E := A_1 \cup \cdots \cup A_k.$$ 

Clearly, $|E| = k$. Since $\mathcal{A}$ satisfies MC, so does the subfamily $(A_1, \ldots, A_k)$. Since $k < n$, it follows by induction that $(A_1, \ldots, A_k)$ has an SDR $(e_1, \ldots, e_k)$. Since $E = A_1 \cup \cdots \cup A_k$ and $|E| = k$ and $e_1, \ldots, e_k$ are distinct, we have $E = \{e_1, \ldots, e_k\}$. It forces that $A_1 = \{e_1\}, \ldots, A_k = \{e_k\}$.

Now consider the family $\mathcal{A}^* := (A_{k+1} \setminus E, \ldots, A_n \setminus E)$ of $n-k$ sets. We claim that $\mathcal{A}^*$ satisfies MC. In fact, for each choice of indices $k+1 \leq j_1 < \cdots < j_l \leq n$, consider the subfamily $(A_1, \ldots, A_k, A_{j_1}, \ldots, A_{j_l})$ of $\mathcal{A}$. We then have 

$$|E \cup A_{j_1} \cup \cdots \cup A_{j_l}| = |A_1 \cup \cdots \cup A_k \cup A_{j_1} \cup \cdots \cup A_{j_l}| \geq k + l.$$ 

Since $|E| = k$, it follows that 

$$|(A_{j_1} \setminus E) \cup \cdots \cup (A_{j_l} \setminus E)| \geq l.$$ 

Thus $\mathcal{A}^*$ satisfies MC. Since $n - k \leq n - 1$, by induction $\mathcal{A}^*$ has an SDR $(e_{k+1}^*, \ldots, e_n^*)$. Then $(e_1, \ldots, e_k, e_{k+1}^*, \ldots, e_n^*)$ is an SDR for $\mathcal{A}$.

**The Room to Spare Case:** For every $k$ with $1 \leq k \leq n - 1$ and every subfamily of $\mathcal{A}$ with $k$ sets, its union contains at least $k+1$ elements. (The union contains more elements that needed for MC, so we have room to spare.) Then
each set $A_i$ with $i \leq n - 1$ contains at least one element, actually two because of room to spare. We take any element $e_n \in A_n$, and set $A'_i := A_i \setminus \{e_n\}$. We claim that the family $\mathcal{A'} := (A'_1, \ldots, A'_{n-1})$ satisfies MC. In fact, for each choice of indices $i_1 < \cdots < i_k$ with $i_1 \geq 1$ and $i_k \leq n - 1$, we have

$$|A'_{i_1} \cup \cdots \cup A'_{i_k}| \geq |A_{i_1} \cup \cdots \cup A_{i_k}| - 1 \geq (k + 1) - 1 = k.$$ 

Hence $\mathcal{A'}$ satisfies MC. By induction, $\mathcal{A'}$ has an SDR $(e_1, \ldots, e_{n-1})$. Then $(e_1, \ldots, e_{n-1}, e_n)$ is an SDR for $\mathcal{A}$.

The Hall theorem can be stated in terms of bipartite graphs.

**Theorem 4.2.** Let $\mathcal{A} = (A_1, A_2, \ldots, A_n)$ be a family of sets. Then the size $m(\mathcal{A})$ of a largest subfamily of $\mathcal{A}$ that has an SDR is given by

$$m(\mathcal{A}) = \min \{n, n - k + |A_{i_1} \cup \cdots \cup A_{i_k}| : i_1 < \cdots < i_k\}.$$ 

**Proof.** Let $G = (X, \Delta, Y)$ be the bipartite graph associated with the family $\mathcal{A}$, where $X = \{1, 2, \ldots, n\}$, $Y = A_1 \cup \cdots \cup A_n$, and

$$\Delta = \{(i, y) \in X \times Y : y \in A_i\}.$$ 

To find the largest size of a subfamily of $\mathcal{A}$ that has an SDR, it is equivalent to finding the matching number of the bipartite graph $G$; and by Lemma 12 it is equivalent to finding the covering number of $G$. Thus $m(\mathcal{A}) = c(G)$.

Let $S$ be a covering of $G$. Let $|S \cap X| = n - k$ and let $\{i_1, i_2, \ldots, i_k\}$ be the complement of $S \cap X$ in $X$. Since the edges $(i_j, a)$, where $a \in A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k}$, can not be covered by $S \cap X$, such edges must be covered by $S \cap Y$. It follows that

$$S \cap Y = A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k}.$$ 

Thus

$$|S| = |S \cap X| + |S \cap Y| = n - k + |A_{i_1} \cup \cdots \cup A_{i_k}|.$$ 

Therefore, the covering number $c(G)$ of $G$ is

$$c(G) = \min \{n, n - k + |A_{i_1} \cup \cdots \cup A_{i_k}| : i_1 < \cdots < i_k\}.$$ 

□
5 Stable Marriages

In the Marriage Problem of $n$ men and $n$ women, we assume that each woman ranks each man in accordance with her preference for that man as a spouse, no tie is allowed. So each woman has a total ordering about the $n$ men. Similarly, each man has a total ordering about the $n$ women. It is clear that there are $n!$ possible ways of complete marriage. A complete marriage is called **unstable** if there exist two women $A$ and $B$ and two men $a$ and $b$ such that

- $A$ and $a$ get married,
- $B$ and $b$ get married,
- $A$ prefers $b$ rather than $a$,
- $b$ prefers $A$ rather than $B$.

A complete marriage is called **stable** if it is not unstable. Does there always exist a stable marriage? If it does, how to find a stable marriage?

Let $G = (X, \Delta, Y)$ be a complete bipartite graph, where $X$ is the set of the $n$ women and $Y$ is the set of the $n$ men; we write

$$X = \{x_1, x_2, \ldots, x_n\}, \quad Y = \{y_1, y_2, \ldots, y_n\}, \quad \Delta = \{(x_i, y_i) : x_i \in X, y_i \in Y\}. $$

Each woman $x_i$ ranks each man $y_j$ by assigning integer $r_{ij}$ between 1 and $n$. Then $(r_{i1}, r_{i2}, \ldots, r_{in})$ is a permutation of $\{1, 2, \ldots, n\}$. Similarly, each man $y_j$ ranks each woman $x_i$ by assigning integer $s_{ji}$ so that $(s_{j1}, s_{j2}, \ldots, s_{jn})$ is a permutation of $\{1, 2, \ldots, n\}$. We thus have the preference ranking matrix

$$M = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ x_1 & (r_{11}, s_{11}) & (r_{12}, s_{21}) & \cdots & (r_{1n}, s_{1n}) \\ x_2 & (r_{21}, s_{12}) & (r_{22}, s_{22}) & \cdots & (r_{2n}, s_{2n}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_n & (r_{n1}, s_{1n}) & (r_{n2}, s_{2n}) & \cdots & (r_{nn}, s_{nn}) \end{bmatrix}.$$ 

**Example 5.1.** For $n = 2$, let the preference ranking matrix be

$$M = \begin{bmatrix} (1, 2) & (2, 2) \\ (2, 1) & (1, 1) \end{bmatrix}. $$
This means that the woman $x_1$ prefers the man $y_1$ than $y_2$, the woman $x_2$ prefers the man $y_2$ than $y_1$. The two women $x_1$ and $x_2$ have opposite taste. However, both $y_1$ and $y_2$ prefers $x_2$ than $x_1$. The men $y_1$ and $y_2$ have the same taste.

There are two possible complete marriages:

$$x_1 \leftrightarrow y_1, \ x_2 \leftrightarrow y_2 \quad \text{and} \quad x_1 \leftrightarrow y_2, \ x_2 \leftrightarrow y_1.$$ 

The first one is stable because the three guys $x_1, x_2, y_2$ are happy, only $y_1$ is upset, but he has no choice. However, the second one is unstable because the wife $x_2$ prefers the man $y_2$ than her current husband $y_1$ and the husband $y_2$ prefers the woman $x_2$ than his current wife $x_1$.

**Example 5.2.** For $n = 3$, let the preference ranking matrix be

$$M = \begin{bmatrix}
(1,3) & (3,2) & (2,1) \\
(2,1) & (1,3) & (3,2) \\
(3,2) & (2,1) & (1,3)
\end{bmatrix}$$

There are $3! = 6$ possible complete marriages. There are two complete stable marriages:

$$x_1 \leftrightarrow y_1, \ x_2 \leftrightarrow y_2, \ x_3 \leftrightarrow y_3 \quad \text{(each woman gets her first choice)}$$

$$x_1 \leftrightarrow y_3, \ x_2 \leftrightarrow y_1, \ x_3 \leftrightarrow y_2 \quad \text{(each man gets his first choice)}$$

The first marriage is stable even each man gets his last choice; the second marriage is stable even each woman gets her second choice. In general letting each woman get her first choice does not necessarily result a complete marriage; for instance, if two women have the same first choice. There are four unstable complete marriages:

$$x_1 \leftrightarrow y_1, \ x_2 \leftrightarrow y_3, \ x_3 \leftrightarrow y_2 \quad \text{($x_2$ and $y_1$ prefer each other)}$$

$$x_1 \leftrightarrow y_3, \ x_2 \leftrightarrow y_2, \ x_3 \leftrightarrow y_1 \quad \text{($x_3$ and $y_2$ prefer each other)}$$

$$x_1 \leftrightarrow y_2, \ x_2 \leftrightarrow y_1, \ x_3 \leftrightarrow y_3 \quad \text{($x_1$ and $y_3$ prefer each other)}$$

$$x_1 \leftrightarrow y_2, \ x_2 \leftrightarrow y_3, \ x_3 \leftrightarrow y_1 \quad \text{($x_1$ and $y_3$ prefer each other)}$$

We now show that a stable marriage always exists and give an algorithm to find a stable marriage. Thus complete chaos can be avoided.
Algorithm 5.1 (Gale-Shapley, 1962). Deferred Acceptance Algorithm for stable marriage of women and men. Begin with all women marked as rejected.

**Step 0.** If there is no rejected woman, then STOP; a stable marriage is obtained. Otherwise, go to **Step 1**.

**Step 1.** Let each woman marked as rejected choose the man whom she ranks highest among all those men who have not yet rejected her. Then go to **Step 2**.

**Step 2.** Let each man pick out the woman he ranks highest among all those women who have chosen him and whom he has not yet rejected, defer to her decision, and reject the others. Then return to **Step 0**.

**Proof.** During the execution of the algorithm, the women propose to men in **Step 1**, some men and some women become engaged in **Step 2**. However, the men engaged are able to break engagements if they receive a better offer in the next run. It is clear that no two men can engage a same woman because of **Step 1**, and no two women can be engaged to a same man because of **Step 2**. We have the following properties:

- Once a man becomes engaged he remains engaged throughout the execution of the algorithm, but his fiancee may be changed. A change of fiancee is an improvement in the eyes of the man who breaks engagement.

- A woman may be engaged or disengaged several times during the execution of the algorithm. However, each new engagement results in a less desirable fiance.

Since each woman cannot be rejected twice by a same man, the number of times to be rejected for each woman is at most the number of men. This indicates that eventually there is no rejected woman, so the algorithm stops after a finite number of runs of **Step 1** and **Step 2**. Whenever there is no rejected woman, no two women had proposed to a same man in **Step 1** in the previous run. Then each man was chosen by one woman, so each man was engaged in **Step 2**. Now we obtain a complete marriage. Next we show that this marriage is stable.
Given two women \(x, x'\) and two men \(y, y'\) such that \(x\) marries \(y\) and \(x'\) marries \(y'\). If \(x\) prefers \(y'\) rather than \(y\), then \(x\) chose \(y'\) at some stage of the algorithm, but \(y'\) rejected \(x\) at the next stage. This means that \(y'\) preferred someone else rather than \(x\) when \(x\) chose him. Since every man gets his fiancee improved during execution of the algorithm, thus \(y'\) prefers \(x'\) rather than \(x\). Thus, it can not be happen that \(y'\) prefers \(x\) rather \(x'\).

\[\square\]

**Example 5.3.** For \(n = 4\), apply the Deferred Acceptance Algorithm to find a stable marriage for the preference ranking matrix

\[
M = \begin{bmatrix}
x_1 & y_1 & y_2 & y_3 & y_4 \\
x_2 & (3, 3) & (1, 4) & (4, 1) & (2, 4) \\
x_3 & (4, 2) & (1, 2) & (2, 3) & (3, 2) \\
x_4 & (4, 1) & (2, 1) & (3, 4) & (1, 3) \\
\end{bmatrix}
\]

Applying Step 1,

\(x_1 \rightarrow y_2, \ x_2 \rightarrow y_2, \ x_3 \rightarrow y_4, \ x_4 \rightarrow y_2\).

Applying Step 2, we obtain a partial marriage

\(x_1 \leftrightarrow \emptyset, \ x_2 \leftrightarrow y_2, \ x_3 \leftrightarrow y_4, \ x_4 \leftrightarrow \emptyset\).

Continuing to apply Step 1 and Step 2,

\(x_1 \rightarrow y_4, \ x_2 \leftrightarrow y_2, \ x_3 \leftrightarrow y_4, \ x_4 \rightarrow y_4; \ x_1 \leftrightarrow \emptyset, \ x_2 \leftrightarrow y_2, \ x_3 \leftrightarrow \emptyset, \ x_4 \leftrightarrow y_4; \ x_1 \rightarrow y_1, \ x_2 \leftrightarrow y_2, \ x_3 \rightarrow y_2, \ x_4 \leftrightarrow y_4; \ x_1 \leftrightarrow y_1, \ x_2 \leftrightarrow \emptyset, \ x_3 \leftrightarrow y_2, \ x_4 \leftrightarrow y_4; \ x_1 \leftrightarrow y_1, \ x_2 \rightarrow y_3, \ x_3 \leftrightarrow y_2, \ x_4 \leftrightarrow y_4; \ x_1 \leftrightarrow y_1, \ x_2 \leftrightarrow y_3, \ x_3 \leftrightarrow y_2, \ x_4 \leftrightarrow y_4.\)

We thus obtain a stable marriage

\(x_1 \leftrightarrow y_1, \ x_2 \leftrightarrow y_3, \ x_3 \leftrightarrow y_2, \ x_4 \leftrightarrow y_4.\)

Among the stable complete marriages we may wish to compare one stable complete marriage with certain other stable complete marriages. A man \(y\) is
called feasible for a woman $x$ provided there is a stable complete marriage in which $y$ is the spouse of $x$. A man is not feasible for a woman is called infeasible for the woman. Obviously, feasibility is symmetric in the sense that if $y$ is feasible to $x$, then $x$ is also feasible to $y$. A stable complete marriage is called optimal for a woman $x$ provided that $x$ ranks her spouse in the marriage higher than or equal to all her feasible spouses. A stable complete marriage is called women-optimal provided it is optimal for each woman. In a similar way we can define a men-optimal stable marriage. It is not obvious whether there exist women-optimal and men-optimal stable complete marriages. In fact, it is not even clear whether it results a complete marriage when each woman is independently given her best feasible spouse. The following theorem clarifies these questions.

**Theorem 5.2.** The stable complete marriage obtained from the Deferred Acceptance Algorithm, with the woman choosing the men, is women-optimal.

**Proof.** The idea is to show that, if a woman $x$ is rejected by a man $y$, the man $y$ is infeasible for $x$. If so, at the end of the Deferred Acceptance Algorithm, each woman obtains as her spouse the man she ranks highest among all the men that are feasible for her, and hence the complete marriage is women-optimal. To this end, we show that at the end of the $k$th round of Algorithm 5.1, all the men who have rejected a woman $x$ are infeasible for $x$. We proceed induction on $k$.

Recall that a man $y$ is infeasible for a woman $x$ if and only if there is no stable complete marriage in which $y$ is the spouse of $x$; in other words, if there is a complete marriage in which $y$ is the spouse of $x$, then the complete marriage is unstable.

For $k = 1$, let $x$ be a woman who is rejected by a man $y$ at the end of the first round. Then $x$ ranks $y$ highest among all men, but $y$ prefers another woman $x'$ who also ranks $y$ highest among all men. If there is a complete marriage such that $x \leftrightarrow y$ and $x' \leftrightarrow y'$, then $x'$ prefers $y$ rather than $y'$ and $y$ prefers $x'$ rather than $x$. This means that the complete marriage is unstable. Now assume it is true for $k$, that is, all men are infeasible for those women whom they reject at or before the end of the $k$th round. Consider the case of $k + 1$.

Let $y$ be a man who rejects a woman $x$ at the end of the $(k + 1)$th round.
We need to show that $y$ is infeasible for $x$. Suppose $y$ is feasible for $x$, this is, there is a stable complete marriage $\mathcal{M}$ in which $y$ is the spouse of $x$. Since the men who did reject $x$ before the $(k+1)$th round are infeasible for $x$, the woman $x$ ranks $y$ highest among all the men feasible for $x$. Since $y$ rejects $x$ in Step 2 of the $(k+1)$th round, the man $y$ chooses another woman $x'$ rather than $x$ in the same Step 2. Note that $x'$ ranks $y$ highest among all the men who did not reject her before the $(k+1)$th round, and by the induction hypothesis, the men who did reject $x'$ before the $(k+1)$th round are infeasible for $x'$. So $x'$ ranks $y$ highest among all the men feasible for $x'$ and some possibly infeasible for $x'$. Let $y'$ be the spouse of $x'$ in $\mathcal{M}$. Then in the complete marriage $\mathcal{M}$, we have 

$$x \leftrightarrow y, \ x' \leftrightarrow y',$$

but $x'$ prefers $y$ rather than $y'$ (since $y'$ is feasible and $y$ is ranked the highest by $x'$ among all the feasible men and some possibly infeasible men for $x'$), $y$ prefers $x'$ rather than $x$ (since $y$ rejects $x$ and accept $x'$). This means that the complete marriage $\mathcal{M}$ is unstable, contrary to that $\mathcal{M}$ is stable.  

Corollary 5.3. In any women-optimal stable complete marriage, each man is paired with the woman he ranks lowest among all the women feasible for him.

Proof. Suppose the theorem is not true, that is, there exist a women-optimal stable marriage $\mathcal{M}$ such that $x \leftrightarrow y$, and a stable complete marriage $\mathcal{M}'$ such tat $x' \leftrightarrow y$, but $y$ ranks $x'$ lower than $x$. Let $y'$ be the spouse of $x$ in the complete marriage $\mathcal{M}'$. The in the stable complete marriage $\mathcal{M}'$, we have 

$$x \leftrightarrow y', \ x' \leftrightarrow y.$$

Since both $y, y'$ are feasible for $x$ (because $\mathcal{M}$ and $\mathcal{M}'$ are stable marriages) and $x$ ranks $y$ highest among all the men feasible for $x$ (because $\mathcal{M}$ is women-optimal), then $x$ prefers $y$ rather than $y'$. Note that $y$ prefers $x$ rather $x'$ by assumption. This means that the complete marriage $\mathcal{M}'$ is unstable, contrary to that $\mathcal{M}'$ is stable.  

Corollary 5.4. The women-optimal and men-optimal stable complete marriages are identical if and only if there is exactly one stable complete marriage.
Proof. If the women-optimal and men-optimal stable complete marriages are identical, then by Corollary 5.3 each woman (man) gets her (his) best and worst partner taken over all stable complete marriages. It follows that there is exactly one stable complete marriage. Conversely, if there is exactly one stable complete marriage, then the women-optimal and men-optimal stable complete marriages must be identically the same.

Example 5.4. Suppose an even number $2n$ of girls wish to pair up as roommates. Each girl ranks the other girls in the order $1, 2, \ldots, 2n - 1$ of preference. A complete marriage in this situation is a pairing of the girls into $n$ pairs. A complete marriage is unstable provided there exist two girls who are not roommates such that each of them prefers the other person’s roommate to her current roommate. A complete marriage is stable if it is not unstable. Does there always exists a stable complete marriage?

Consider the case of four girls $a, b, c, d$; each of the four girls ranks the others as the following

\[
\begin{align*}
    a & : b, c, d \\
    b & : c, a, d \\
    c & : a, b, d \\
    d & : \text{any order}
\end{align*}
\]

We shall see that there is no stable complete marriage. Suppose $a$ and $d$ are roommates, then $b$ and $c$ would be roommates. However, $a$ prefers $c$ rather $d$ and $c$ prefers $a$ rather $b$. So the complete marriage

\[
a \leftrightarrow d, \ b \leftrightarrow c
\]

is unstable. Similar, the complete marriages

\[
\begin{align*}
    b \leftrightarrow d, \ a \leftrightarrow c \quad \text{and} \quad c \leftrightarrow d, \ a \leftrightarrow b
\end{align*}
\]

are also unstable. So there is no stable complete marriage.