1 Hyperbolic geometry

János Bolyai (1802-1860), Carl Friedrich Gauss (1777-1855), and Nikolai Ivanovich Lobachevsky (1792-1856) are three founders of non-Euclidean geometry.

Hyperbolic geometry is, by definition, the geometry that assume all the axioms for neutral geometry and replace Hilbert’s parallel postulate by its negation, which is called the hyperbolic axiom.

Hyperbolic axiom (Negation of Hilbert axiom). There exists a line \( l \) and a point \( P \) not on \( l \) such that at least two distinct lines parallel to \( l \) pass through \( P \).

**Theorem 1.1.** In hyperbolic geometry, all triangles have angle sum less than 180°, and all convex quadrilaterals have angle sum less than 360°. In particular, there is no rectangle.

*Proof. Trivial.*

**Theorem 1.2** (Universal Hyperbolic Theorem). In hyperbolic geometry, for every line \( l \) and every point \( P \) not on \( l \) there pass through \( P \) at least two distinct lines parallel to \( l \). In fact there are infinitely many lines parallel to \( l \) through \( P \).

*Proof.* Drop segment \( PQ \) perpendicular to \( l \) with foot \( Q \) on \( l \). Erect line \( m \) at \( P \) perpendicular to \( PQ \). Then \( l, m \) are parallel. Pick a point \( R \) on \( l \) other than \( Q \), and erect line \( t \) at \( R \) perpendicular to \( l \). Drop line \( n \) through \( P \) perpendicular to \( t \), intersecting \( t \) at \( S \). If \( S \) is on \( m \), then \( S \) is the intersection \( m \) and \( t \), and subsequently \( \square PQRS \) is a rectangle, which is impossible in hyperbolic geometry. So point \( S \) is not on \( m \). Hence \( m, n \) are distinct lines through \( P \), both are parallel to \( l \). See Figure 1.

Let \( R' \) be point on \( l \) other than \( Q, R \), and \( t' \) be line through \( R' \) perpendicular to \( l \). There exists line \( n' \) through \( P \) perpendicular to \( t' \), intersecting \( t' \) at \( S' \). If \( PS = PS' \), then \( \square RR'S'S \) is a rectangle, which is impossible. So \( PS, PS' \) are distinct lines. Thus for all points \( R \) on \( l \) other than \( Q \), the lines \( PS \) through \( P \) perpendicular to \( l \) are all distinct.  

![Figure 1: Existence of infinite parallels](image-url)
Definition 1. Two triangles are $\Delta ABC$ and $\Delta A'B'C'$ said to be similar if their vertices can be put in on-to-one correspondence so that corresponding angles are congruent, i.e., $\angle A, \angle B, \angle C$ are congruent to angles $\angle A', \angle B', \angle C'$ respectively.

Theorem 1.3 (AAA criterion for congruence of hyperbolic triangles). In hyperbolic geometry, if two triangles are similar then they are congruent.

![Figure 2: Similar triangles are congruent in hyperbolic geometry](image)

Proof. Given similar triangles $\Delta ABC$ and $\Delta A'B'C'$. Suppose the statement is not true, i.e., $\Delta ABC$ is not congruent to $\Delta A'B'C'$. Then $AB \not\approx A'B'$, $AC \not\approx A'C'$, $BC \not\approx B'C'$; otherwise $\Delta ABC \cong \Delta A'B'C'$ by ASA. We may assume $AB < A'B'$ and $AC < A'C'$. Lay off segment $AB$ on ray $r(A', B')$ to have point $B''$ such that $AB \cong A'B''$ and $A' * B'' * B'$, and lay off segment $AC$ on ray $r(A', C')$ to have point $C''$ such that $AC \cong A'C''$ and $A' * C'' * C'$. See Figure 2. Then $\Delta ABC \cong \Delta A'B''C''$ by SAS. Hence

$$\angle A'B''C'' \cong \angle B \cong \angle B', \quad \angle A'C''B'' \cong \angle C \cong \angle C'. \tag{1}$$

It follows that lines $\overline{B''C''}, \overline{B'C'}$ are parallel because of Congruent Corresponding Angles. So $\square B'C'C''B''$ is a convex quadrilateral. Since angles $\angle A'B''C'', \angle B'B''C''$ are supplementary, angles $\angle A'C''B'', \angle C'C''B''$ are supplementary, and $\angle B' \cong \angle A'B''C'', \angle C' \cong \angle A'C''B''$, then the angle sum of $\square B'C'C''B''$ is 360°. This is a contradiction. \(\square\)

Theorem 1.3 says that in hyperbolic geometry it is impossible to magnify or shrink a triangle without distortion. So in hyperbolic world photography would be inherently surrealist.

Another consequence of Theorem 1.3 is that the length of a segment may be determined by angles in hyperbolic geometry. For example, an angle of an equilateral triangle determines the length of a side uniquely. This fact is sometimes referred to that hyperbolic geometry has an absolute unit length.

2 Parallels that admit a common perpendicular

Given lines $l, l'$ and points $A, B, C, \ldots$ on $l$. Drop perpendiculars $AA', BB', CC', \ldots$ from $A, B, C, \ldots$ to $l'$ with feet $A', B', C', \ldots$ on $l'$ respectively. We say that $A, B, C, \ldots$ are equidistant from $l'$ if all these perpendicular segments are congruent to one another. See Figure 3.

Theorem 2.1 (At most two points equidistant). Given two distinct parallels $l, l'$ in hyperbolic geometry. Then any set of points on $l$ equidistant from $l'$ contains at most two points.

Proof. Suppose it is not true, i.e., there is a set of three points $A, B, C$ on $l$ equidistant from $l'$. Then quadrilaterals $\square AB'B'A', \square ACC'A', \square BCC'B'$ are Saccheri quadrilaterals (the base
angles are right angles and the sides are congruent). Then the summit angles of the Saccheri quadrilaterals are congruent, i.e.,
\[
\angle BAA' \cong \angle ABB', \quad \angle CAA' \cong \angle ACC', \quad \angle CBB' \cong \angle BCC'.
\]
Thus \(\angle ABB' \cong \angle CBB'\). Since \(\angle ABB', \angle CBB'\) are supplementary, they must be right angles. Hence all \(\square ABB'A', \square ACC'A', \square BCC'B'\) are rectangles, which is impossible. \(\square\)

**Lemma 2.** Given a Saccheri quadrilateral \(\square ABB'A'\) with base right angles \(\angle A', \angle B'\) and equal opposite sides \(AA', BB'\). Let \(M, M'\) be the middle points of \(AB, A'B'\) respectively. Then segment \(MM'\) is perpendicular to both lines \(\overline{AB} \) and \(\overline{A'B'}\).

**Proof.** Draw segments \(A'M\) and \(B'M\). Note that \(AA' \cong BB'\), \(\angle A \cong \angle B\), and \(AM \cong BM\). Then \(\triangle A'AM \cong \triangle B'MM'\) by SAS. So \(A'M \cong B'M\). Hence \(\triangle A'MM' \cong \triangle B'MM'\) by SSS. We then have \(\angle A'M'M \cong \angle B'M'M\). Subsequently, \(\angle A'M'M\) and \(\angle B'M'M\) are right angles. So \(MM'\) is perpendicular to the base \(A'B'\). Note that \(\angle A'M'M \cong \angle B'M'M\) and \(\angle A'MA' \cong \angle B'MB'\). Then \(\angle A'M'M \cong \angle B'M'M\) by angle addition. Subsequently, \(\angle A'M'M\) and \(\angle B'M'M\) are right angles. So \(MM'\) is perpendicular to the summit \(AB\). \(\square\)

**Theorem 2.2** (Divergent and symmetric parallels). Let \(l, l'\) be two lines perpendicular to a segment \(MM'\) with \(M \in l, M' \in l'\).

(a) Then \(|MM'| < |XY'|\) for all \(X \in l, Y' \in l'\) with \(XY' \neq MM'\).

(b) If \(M\) is the middle point of a segment \(AB\) on \(l\), then \(A, B\) are equidistant from \(l'\).

(c) If \(M \ast B \ast D\) on \(l\) and \(BB', DD'\) are segments perpendicular to \(l'\) with feet \(B', D' \in l'\), then \(BB' < DD'\).

**Proof.** (a) It is clear that \(MM' < YY'\) for all \(Y'\) on \(l'\) with \(Y' \neq M'\). Let \(X\) be a point on \(l\) with \(X \neq M\). Let \(XX'\) be segment perpendicular to \(l'\) with foot \(X'\) on \(l'\). Then \(\square MM'XX'\) is a Lambert quadrilateral. Thus
\[
MM' < XX'
\]
by properties of Lambert quadrilaterals. Since \(XX' < YY'\) for \(Y'\) on \(l'\) with \(Y' \neq X'\). We see that \(MM' < YY'\).

(b) Let \(AA', BB'\) be segments perpendicular to \(l'\) with \(A', B' \in l'\). Draw segments \(AM'\) and \(BM'\). Then \(\triangle A'MM' \cong \triangle B'MM'\) by SAS. So \(AM' \cong BM'\) and \(\angle A'M'M \cong \angle B'M'M\).
Subsequently, ∠AM′A’ ≅ ∠BM′B’ by angle subtraction. Thus ΔAA′M′ ≅ ΔBB′M′ by SAA. Hence AA′ ≅ BB′ and A′M′ ≅ B′M′.

(c) Note that □M′B′BM and □M′D′DM are Lambert quadrilaterals. Then ∠B′BM and ∠D′DM are acute angles. So ∠B′BD is obtuse, for it is supplementary to ∠B′BM. Hence ∠D′DB = ∠D′DM < ∠B′BD. Therefore BB′ < DD′ by the property of □B′D′DB.

\[\text{Proposition 2.3} \quad \text{(Asymptotic and monotonic parallels). Given parallels} \ l, l' \ \text{in hyperbolic geometry, no two points of} \ l \ \text{are equidistant from} \ l'. \ \text{Let} \ AA', BB', CC' \ \text{be perpendicular segments to} \ l' \ \text{with} \ A* B* C \ \text{on} \ l \ \text{and} \ A', B', C' \ \in \ l'. \ \text{See Figure 6.}
\]

\text{(a) If} AA' < BB', \ \text{then} BB' < CC'.
\text{(b) If} BB' < CC', \ \text{then} AA' < BB'.

\[\text{Figure 6: Monotone distance between asymptotic parallels}\]

\text{Proof.} \ \text{Consider quadrilaterals} \ □A'B'BA \ \text{and} \ □B'C'CB.

\text{(a) Since} AA' < BB', \ \text{then} ∠ABB' < ∠BAA'. \ \text{Since the angle sum of} \ □A'B'BA \ \text{is less than} 360°, \ \text{it follows that} ∠ABB' \ \text{is acute}. \ \text{So} ∠B'B'C' \ \text{must be acute, since the angle sum of} \ □B'C'CB \ \text{is less than} 360°. \ \text{Of course} ∠BCC' < ∠B'B', \ \text{subsequently,} BB' < CC' \ \text{by the property of} \ □B'C'CB.

\text{(b) Fix a point} D \ \text{on} \ l \ \text{with} \ A* B* C* D. \ \text{For each} \ X \ \text{on the open ray} \ r(D, A) \ \text{we write} \ \left|DX = x \right| \ \text{and define} \ f(x) = |XX'|, \ \text{where} \ XX' \ \text{is perpendicular to} \ l' \ \text{with foot} \ X' \ \text{on} \ l'. \ \text{We claim that} f(x) \ \text{is a continuous function for} \ x > 0. \ \text{In fact, fix an} x_0 \ \text{with point} X_0 \ \text{on} \ l \ \text{such that} |BX_0| = x_0. \ \text{Let} X_0X_0' \ \text{be segment perpendicular to} \ l' \ \text{with} \ X_0' \ \in \ l'. \ \text{Note that}

\[|XX'| \leq |XX_0'| \leq |X_0X_0'| + |XX_0|, \ |X_0X_0'| \leq |X_0X'| \leq |XX'| + |XX_0|.
\]

Then

\[|f(x) - f(x_0)| = \begin{cases} \ |XX'| - |X_0X_0'| & \text{if} |XX'| \geq |X_0X_0'| \\ |X_0X_0'| - |XX'| & \text{if} |XX'| < |X_0X_0'| \end{cases} \leq |XX_0'| = |x - x_0|.
\]

Clearly, \ f(x) \ \text{is continuous at} x_0. \ \text{So} f(x) \ \text{is a continuous function for} x > 0.

\text{Suppose} AA' > BB'. \ \text{Note that} AA' \ \not\equiv CC'. \ \text{If} |AA' | > |CC'|, \ \text{by intermediate value theorem there exists a} Y \ \text{with} B* Y* C \ \text{such that} |YY'| = |AA'|. \ \text{Then} A, Y \ \text{are equidistant}
from \(l'\), which is impossible. If \(|AA'| > |CC'|\), by intermediate value theorem there exists a point \(Z\) with \(A \ast Z \ast B\) such that \(|ZZ'| = |CC'|\). Then \(C, Z\) are equidistant from \(l'\), which is impossible.

We then must have \(AA' < BB'\). \(\square\)

3 Limiting parallel rays

Given a line \(l\) in hyperbolic geometry and a point \(P\) not on \(l\). Let \(m\) be a line through \(P\) parallel to \(l\) with left ray \(r(P, R)\). Drop perpendicular segment \(PQ\) to \(l\) with foot \(Q\) on \(l\). We consider rays between \(r(P, Q)\) and \(r(P, R)\), and want to find the critical ray \(r(P, X)\), called the **left limiting parallel ray to \(l\) through \(P\)\), that does not meet \(l\) but any ray between \(r(P, X)\) and \(r(P, Q)\) meets \(l\). Likewise, there is a **right limiting parallel ray to \(l\) through \(P\)** on the opposite side of \(PQ\). See Figure 7.

**Theorem 3.1.** Given a line \(l\) and a point \(P\) not on \(l\) in hyperbolic geometry. Let \(PQ\) be segment perpendicular to \(l\) with foot \(Q\) on \(l\). Then there exist two non-opposite rays \(r(P, X), r(P, X')\) on opposite sides of line \(PQ\), satisfying the properties:

(a) Each of rays \(r(P, X), r(P, X')\) does not meet \(l\).

(b) A ray \(r(P, Y)\) meets \(l\) if and only if it is between \(r(P, X)\) and \(r(P, X')\).

(c) \(\angle QPX \cong \angle QPX'\).

**Proof.** Let \(m\) be the line through \(P\) perpendicular to \(PQ\). Pick a point \(R\) on the left side of \(m\) and a point \(R'\) on the right side of \(m\) separated by \(P\). Draw segments \(QR\) and \(QR'\). Then all rays between \(r(P, Q)\) and \(r(P, R)\) inclusive are represented by \(r(P, Y)\) with \(Y \in QR\). See Figure 7.

![Figure 7: Limiting parallel rays](image-url)

(a) Let \(\Sigma_1\) be the set of points \(Y \in r(Q, R)\) such that the ray \(r(P, Y)\) does not meet \(l\), and \(\Sigma_2\) the complement of \(\Sigma_1\) in \(QR\). It is easy to see that both \(\Sigma_1, \Sigma_2\) are convex. So \(\Sigma_1, \Sigma_2\) form a Dedekind cut of \(QR\). Then there exists a unique point \(X \in QR\) such that \(\Sigma_1, \Sigma_2\) are two rays (one of them is an open ray) of \(QR\) separated by \(X\). We claim that \(X \in \Sigma_1\).

Suppose \(X \in \Sigma_2\), i.e., \(r(P, X)\) meets \(l\) at \(S\). Pick a point \(T\) on \(l\) such that \(T \ast S \ast Q\). Then ray \(r(P, T)\) is between \(r(P, R)\) and \(r(P, X)\). So \(r(P, T)\) meets \(RQ\) at \(U\) and \(R \ast U \ast X\), i.e., \(U \in \Sigma_2\), which is a contradiction. The existence of ray \(r(P, X')\) is analogous.

(b) Since \(R \in \Sigma_1\) and \(Q \in \Sigma_2\), we see that \(R \ast X \ast Q\). It is obvious that if a ray \(r(P, Y)\) is contained in the open half-plane opposite to \(\tilde{H}(m, Q)\) then \(r(P, Y)\) does not meet \(l\). We then see that a ray \(r(P, Y)\) meets \(l\) if and only if \(r(P, Y)\) is between \(r(P, X)\) and \(r(P, X')\).

(c) Suppose that \(\angle XPQ\) is not congruent to \(\angle X'PQ\), say, \(\angle XPQ < \angle X'PQ\). Find point \(V'\) on \(l\) such that \(r(P, V')\) is between \(r(P, Q)\) and \(r(P, X')\), and \(\angle QPV' \cong \angle QPX\). Mark a point \(V\) on \(l\) such that \(V \ast Q \ast V'\) and \(QV \cong QV'\). Then \(\Delta PQV \cong \Delta PVQ'\) by SAS. So \(\angle QPV \cong \angle QPV' \cong \angle QPX\), i.e., \(r(P, X)\) meets \(l\) at \(V\), which is a contradiction. \(\square\)
The angle $\angle QPX$ is called the **angle of parallelism** at point $P$ with respect to $l$, its degree measure is denoted $\Pi(PQ)^\circ$. We have

$$\Pi(PQ)^\circ < 90^\circ.$$ 

## 4 Classification of parallels

**Theorem 4.1.** Given parallel lines $l, l'$ in hyperbolic geometry.

(a) If $l$ contains a limiting parallel ray to $l'$, then $l, l'$ are asymptotic parallels.

(b) If $l$ does not contain limiting parallel ray to $l'$, then $l, l'$ are divergent parallels.

*Proof.* Fix a point $P$ not on $l'$ and drop a perpendicular $PQ$ to $l'$ with foot $Q \in l'$. Let $m$ be the line through $P$ perpendicular to $PQ$. Pick a point $R$ on $m$ other than $P$. Let $r(P, X)$ be a limiting parallel ray to $l'$ with $X \in QR$. See Figure 8.

(a) Let $A, B, C, D$ be points on $l$ with $A \ast B \ast P \ast C \ast D$ and $A, B \in r(P, X)$. Let $AA', BB', CC', DD'$ be segments perpendicular to $l'$ with feet $A', B', C', D' \in l'$. Note that $\angle XPQ$ is acute, $\angle CPQ$ is obtuse, and the angle sum of $\square PQC'C$ is less than $360^\circ$. Then $\angle PCC'$ is acute. Of course $\angle PCC' < \angle CPQ$. So $PQ < CC'$ by property of quadrilaterals with two base right angles.

Analogously, $\angle PDD'$ is acute and $\angle DCC'$ is obtuse. Of course $\angle PDD' < \angle DCC'$. Then $CC' < DD'$ by property of quadrilaterals with two base right angles.

We claim $|AA'| \leq |PQ|$ for all $A$ on open ray $r(P, X)$. Suppose $|AA'| > |PQ|$. Let $S$ be a point on $AA'$ such that $|AS| = \frac{1}{2}(|AA'|-|PQ|)$. Clearly, $|SA'| > |PQ|$. Then $\angle A'SP < \angle XPQ$ by property of quadrilateral with two base right angles. Of course $\angle A'SP$ is acute. Since $r(P, S)$ is between $r(P, Q)$ and $r(P, X)$, the ray $r(P, S)$ meets $l'$ at $T$. Note that $\angle A'ST$ is acute. So $\angle A'SP$ is obtuse, contradicting to that $\angle A'SP$ is acute.

We further claim $AA' < BB'$ for two points $A, B$ on closed ray $r(P, X)$ with $A \ast B \ast C$. Suppose $|AA'| \geq |BB'|$. There exists a point $E$ on $BP$ (maybe $B = P$) such that $AA' \cong EE'$ by continuity of distance function. Let $M, M'$ be the middle points of $AE, A'E'$ respectively. Then $l, l'$ are divergent parallels. Let $F$ be on $l$ such that $F \ast M \ast C$ and $MF \cong MC$. We have $|FF'| = |CC'| > |PQ|$, which is a contradiction.

(b) Assume that $l$ does not contain any limiting parallel ray. If $l = m$, then $l, l'$ are already divergent parallels. If $l \neq m$, we may assume that a ray $r(P, Y)$ of $l$ is between $r(P, R)$ and $r(P, X)$, where $R \ast Y \ast X$. It is easy to see that $PQ < CC' < DD'$ by similar arguments.

Since $\angle XPY$ is acute, by Aristotle's axiom there exists a point $A$ on $r(P, Y)$ such that $AE > PQ$, where $AE$ is perpendicular to $r(P, X)$ with foot $E \in r(P, X)$. Of course $AA' >
AF > AE. So $AA' > PQ$. Thus $l, l'$ cannot be asymptotic (monotonic) parallels. So $l, l'$ must be divergent (symmetric) parallels. \qed

Let $A, A'$ be two distinct points on the same side of a line $AB$ such that lines $AA', BB'$ are parallel. Then the figure, consisting of the segment $AB$ (called the base) and the rays $r(A, A')$ and $r(B, B')$ (called the sides), is called a biangle with vertices $A$ and $B$, denoted $\triangle A'ABB'$. See Figure 10. The interior of biangle $\triangle A'ABB'$ is

\[ \triangle A'ABB' := \triangle A'AB \cap \triangle ABB'. \]

If $P \in \triangle A'ABB'$, either of rays $r(A, P), r(B, P)$ is called an interior ray of \( \triangle DABC \). If each interior ray $r(A, P)$ intersects $r(B, B')$, we say that $r(A, A')$ is limiting parallel to $r(B, B')$ and that biangle $\triangle A'ABB'$ is closed at $A$, written $r(A, A') \parallel r(B, B')$.

**Lemma 3.** Let $\triangle A'ABB'$ be a biangle. See Figure 10.

(a) If $D \ast A \ast A'$, then $r(D, A') \parallel r(B, B')$ if and only if $r(A, A') \parallel r(B, B')$.

(b) If $r(A, A') \parallel r(B, B')$, so is $r(B, B') \parallel r(A, A')$.

**Proof.** (a) Assume $r(D, A') \parallel r(B, B')$. Take a point $P$ in the interior of $\triangle A'ABB'$. It is clear that $P$ is an interior point of biangle $\triangle A'DBB'$. Then $r(D, P)$ meets $r(B, B')$ at $F$ since $\triangle A'DBB'$ is closed at $D$. Note that $P$ is an interior point of $\triangle BAF$. Then $r(A, P)$ is between $r(A, B)$ and $r(A, F)$. Thus $r(A, P)$ meets $BF$ at $G$ with $B \ast G \ast C$. By definition $r(A, A') \parallel r(B, B')$.

Conversely, assume $r(A, A') \parallel r(B, B')$. For each ray $r$ between $r(D, B)$ and $r(D, A')$, we have $r$ meeting $AB$ at $E$ between $A$ and $B$. Pick a point $P$ on $r$ such that $D \ast E \ast P$. Note that $\triangle A'AB > \triangle AED \cong \triangle BEP$. There is a ray $r(A, Q)$ such that $\triangle BAQ \cong \triangle BEP$. Then $r(A, Q) \parallel r(E, P)$. Since $r(A, Q)$ meets $r(B, B')$, we see that $r(E, P)$ must meet $r(B, B')$, i.e., $r(D, P)$ meet $r(B, B')$. Hence $\triangle A'DBB'$ is closed at $D$.

(b) Given an interior point $P \in \triangle ABB'$ and consider the ray $r(B, P)$. Suppose $r(B, P)$ does not meet $r(A, A')$. By the corollary of Aristotle’s axiom there exists a point $Q$ on $r(B, P)$ such that $\triangle AQB < \triangle PBB'$. See Figure 11. Note that $r(A, Q)$ meets $r(B, B')$ at $C$. Then we have triangle $\triangle BCQ$. Thus $\triangle AQB > \angle QBC = \angle PBB'$, which contradicts...
\[ \angle AQB < \angle PBB'. \] So \( r(B, P) \) must meet \( r(A, A') \). Hence \( \angle A'ABB' \) is closed at \( B \), i.e., \( r(B, B') \parallel r(A, A') \).

**Proposition 4.2** (Transitivity of limiting parallelism). If both rays \( r(A, A') \) and \( r(B, B') \) are limiting parallel to ray \( r(C, C') \), then \( r(A, A') \) and \( r(B, B') \) are limiting parallel to each other.

**Proof.**

*Case 1.* Lines \( \overline{AA'} \) and \( \overline{BB'} \) are on opposite sides of line \( \overline{CC'} \). See Figure 12.

It is clear that \( r(A, A') \) and \( r(B, B') \) have no point in common. Let \( AB \) meet \( \overline{CC'} \) at \( D \). We may assume \( C * D * C' \). Now for each point \( P \) interior to \( \angle A'AB \), the ray \( r(A, P) \) meets \( r(C, C') \) at \( E \) since \( r(A, A') \parallel r(C, C') \). We may assume \( C * E * C' \). Then \( r(E, E') \) meets \( r(B, B') \) at \( F \) since \( r(C, C') \parallel r(B, B') \), where \( E * E' * F \). Hence \( r(A, A') \) meets \( r(B, B') \) at \( F \). Therefore by definition \( r(A, A') \) and \( r(B, B') \) are limiting parallel to each other.

*Case 2.* Lines \( \overline{AA'} \) and \( \overline{BB'} \) are on the same side of line \( \overline{CC'} \).

We first claim that \( \overline{AA'} \) and \( \overline{BB'} \) do not meet. Suppose \( \overline{AA'} \) and \( \overline{BB'} \) meet at point \( D \). We may assume that \( D \) belongs to both rays \( r(A, A') \), \( r(B, B') \), and assume \( A * D * A' \), \( B * D * B' \). Take a point \( D' \) such that \( A * D * D' \). Then \( r(D, D') \) meets \( r(C, C') \) since \( r(D, B') \parallel r(C, C') \), i.e., \( r(A, A') \) meets \( r(C, C') \), which is a contradiction. See Figure 13.

Let \( AC \) meet \( \overline{BB'} \) at point \( D \). We may assume \( B * D * B' \). For each point \( P \) interior to \( \angle A'AC \), the ray \( r(A, P) \) meets \( r(C, C') \) at point \( E \). Since \( r(B, D) (= r(B, B')) \) meets the triangle \( \triangle DCE \), the ray \( r(B, B') \) meets either \( AE \) or \( CE \). Since \( r(B, B') \) does not meet \( r(C, C') \), so \( r(B, B') \) meet \( AE \) at \( F \) such that \( A * F * E \). For point \( P \) interior to \( \angle BAD \), the ray \( r(A, P) \) meets \( BD \) between \( B \) and \( D \), of course \( r(A, P) \) meets \( r(B, B') \). Hence \( r(A, A') \) and \( r(B, B') \) are limiting parallel to each other. \( \Box \)
Two rays \( r, s \) are said to be **limiting parallel**, denoted \( r \parallel s \), if \( r \subset s \) or \( s \subset r \) or \( r \parallel s \). Then \( \parallel \) is an equivalence relation on rays in hyperbolic geometry. An equivalence class of rays is called an **ideal point** or **end**, viewing it lying on each ray contained in the equivalence class. Since a point on a line separates the line into two opposite rays, and opposite rays are not equivalent, we see that every line has two ends on it.

If \( A, B \) are vertices of two rays \( r, s \) with \( r \parallel s \). Let \( \mathcal{R} \) denote the ideal point determined by these rays, i.e., \( \mathcal{R} = [r] = [s] \). We write \( r = A\mathcal{R} \) and \( s = B\mathcal{R} \) and refer to the closed biangle with side \( r, s \) as a **singly asymptotic triangle** \( \Delta AB\mathcal{R} \). We shall see that these triangles have some properties in common with ordinary triangles.

**Lemma 4.** In hyperbolic geometry if two lines \( l, m \) are cut by a line \( t \) such that the alternate interior angles are congruent, then \( l, m \) are divergent parallels.

**Proposition 4.3.** Let \( \Delta AB\mathcal{R} \) be a singly asymptotic triangle with a single ideal point \( \mathcal{R} \). Then the exterior angles at \( A, B \) are greater than their respective opposite interior angles, i.e., \( \angle A < \text{ext } \angle B \).
Proof. Extend $AB$ to $C$ such that $A \ast B \ast C$. Draw ray $r(B, D)$ such that $\angle CBD \cong \angle BAA'$ and extend $DB$ to $E$ such that $E \ast B \ast D$. Then $\angle ABE \cong \angle CBD \cong \angle BAA'$. Thus lines $BD, AA'$ are divergent parallels. Since $r(B, B') \parallel r(A, A')$, we see that $r(B, D) \neq r(B, B')$. If $r(B, D)$ is between $r(B, B')$ and $r(B, A)$, then $r(B, D)$ meets $r(A, A')$, which is a contradiction. So we must have $r(B, D)$ between $r(B, C)$ and $r(B, B')$. This means that $\angle CBD < \angle CBB'$, i.e., $\angle CBB' > \angle BAA'$. \qed