Turbulence spectrum of a passive temperature field: Results of a numerical simulation

J. Chasnov,[a] V. M. Canuto,[b] and R. S. Rogallo[c]
NASA Goddard Space Flight Center, Institute for Space Studies, 2880 Broadway, New York, New York 10025

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The spectrum of a passive temperature field, $G(k)$, has been determined by a numerical simulation using three kinds of isotropic turbulent velocity fields. For a time independent and Gaussian velocity field, the resulting $G(k)$ has the form $G(k) = G_0\epsilon_\theta\epsilon^{2/3}\chi^{-3}k^{-17/3}$, with $G_0 = 0.33 \pm 0.02$ Ko, confirming the prediction of Batchelor, Howells, and Townsend [J. Fluid Mech. **5**, 134 (1959)]. For a velocity field developed through the Navier–Stokes equations and then frozen in time, $G(k)$ has the same form as above, but with $G_0 = 0.39 \pm 0.03$ Ko. Finally, for a velocity field developed concurrently with the temperature field, $G(k)$ collapses onto the spectrum obtained using a frozen, developed velocity field only for high enough values of the conductivity χ. For lower values of χ, the power law behavior of $G(k)$ is less clear.

The determination of the universal turbulence spectrum of a passive scalar field has been a controversial problem for more than 30 years. In the first papers on the subject, Obukhov[1] and Corrsin[2] generalized Kolmogorov's universal equilibrium theory to a scalar field. In 1941, Kolmogorov[3] had argued that for very high Reynolds number flows, there should exist a region in k space, the *inertial subrange*, where the three-dimensional energy spectral function $E(k)$ becomes independent of the specific details of the source and is still not directly affected by molecular viscosity. He determined the function $E(k)$ to have the following universal form:

$$E(k) = \text{Ko}\,\epsilon^{2/3}k^{-5/3}, \qquad (1)$$

where ϵ is the energy dissipation rate and Ko is the Kolmogorov constant. It was argued by Obukhov and Corrsin that an analogous universal relation should also hold for a passive temperature field with spectral function $G(k)$, i.e., that in the region called the *inertial-convective subrange*,

$$G(k) = \text{Ba}\,\epsilon_\theta\epsilon^{-1/3}k^{-5/3}, \qquad (2)$$

where ϵ_θ and $G(k)$ are defined by

$$\epsilon_\theta = 2\chi\int_0^\infty k^2G(k)dk,\quad \langle\theta^2\rangle = \int_0^\infty G(k)dk, \qquad (3)$$

χ is the thermal conductivity, $\langle\theta^2\rangle$ is the mean square temperature fluctuation, and Ba is the Batchelor constant. Although intermittent spatial fluctuations in the rates of dissipation or conduction may modify Eqs. (1) and (2), the modifications must be slight since the $k^{-5/3}$ power laws have been strikingly confirmed by experiment.[4]

In 1959, Batchelor[5] and Batchelor, Howells, and Townsend[6] (BHT) attempted to extend Eq. (2) to regions of k space where the direct effects of molecular viscosity ν or thermal conductivity χ become larger than their turbulent analogs ν_t and χ_t. The region where $\nu \gg \nu_t$, but $x \ll x_t$, is called the *viscous-convective subrange*; the region where $\nu \ll \nu_t$, but $\chi \gg \chi_t$, is called the *inertial-conductive subrange*. Conventionally, a value of the Prandtl number $\sigma \equiv \nu/\chi = 1$ is taken to mark the division of these two regions. Recently,[7] it has been argued that the correct dividing value should be σ_t, the turbulent Prandtl number, where $\sigma_t \equiv \nu_t/\chi_t$. In the inertial-convective subrange, σ_t is independent of wavenumber and has the value[7,8] $\sigma_t = \text{Ba}/\text{Ko} \approx 0.5$.

The BHT result in the inertial-conductive subrange is

$$G(k) = G_0\epsilon_\theta\epsilon^{2/3}\chi^{-3}k^{-17/3}, \qquad (4)$$

where the numerical constant G_0 was determined to be $G_0 = \frac{1}{3}$ Ko. The BHT result may be shown to be a consequence of a quasinormal approximation for the nonlinear transfer.[9,10] It has also been derived by Kraichnan using the Lagrangian-history direct-interaction approximation (LHDIA).[11] However, other theoretical arguments and closure approximations have derived spectral functions in the inertial-conductive subrange different from that of BHT. In 1968 Gibson[12,13] presented the case for an inertial-conductive subrange power law of k^{-3}. Other theoretical predictions for this subrange include an exponential form[14] and a power law[15] of $k^{-13/3}$. Recently there have been two more predictions of a $k^{-17/3}$ power law, Eq. (4), but with different values for G_0. Qian[16] has derived a value of G_0 greater than that of BHT by a factor of 3.6, while Canuto, Goldman, and Chasnov[7] have found $G_0 = \frac{8}{27}\,\text{Ba}^{-2}$.

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Given the variety of theoretically predicted power laws, i.e., exponential, $-17/3$, $-13/3$, and -3, it may be hoped that the correct form of the spectrum could be decided experimentally. Unfortunately, the exotic and toxic nature of low Prandtl number materials makes experimentation difficult. Rust and Sesonske[17] have reported measurements of temperature fluctuations for $\sigma = 0.025$, which Gibson[13] claims can best be fitted by k^{-3} rather than $k^{-17/3}$. On the other hand, data by Granatstein *et al.*[18] can be fitted by $k^{-13/3}$, although the Reynolds number may be too low to check theories of universality. Clay's[19] experimental data for liquid Hg ($\sigma = 0.018$) shows some indication of k^{-3} and $k^{-17/3}$ subranges, although his results must be considered inconclusive.

In the absence of definitive theoretical or experimental results, the remaining possibility is to perform a numerical "experiment" to determine the correct spectral function. However, with existing computers, complete numerical resolution of all important scales of motion is restricted to low Reynolds number flows. Kerr[20] through a direct numerical simulation, was able to identify a small Kolmogorov inertial subrange. However, the Reynolds number was too low for the observance of the inertial-convective and inertial-conductive subranges, while the resolution of the scalar was too poor for the observance of the viscous-convective subrange. Recently,[21] through the use of a high symmetric flow, an inertial range spectrum was successfully simulated over almost one decade of wavenumber. Results for the passive scalar were not reported.

In this Letter, we present the results of the first numerical simulation of the inertial-conductive subrange. In light of the limitations placed on a direct numerical simulation of the Navier–Stokes equations because of computer speed and memory requirements, we do not attempt to fully resolve the velocity field. Instead, our approach is motivated by a theoretical argument found in the BHT paper.[6] In the inertial range of the velocity field, the characteristic time scale of velocity fluctuations is $\tau_v \sim (\epsilon k^2)^{-1/3}$, while in the conductive subrange of the scalar field, the scalar fluctuations are damped out in a time of the order of $\tau_\theta \sim (\chi k^2)^{-1}$. It has been argued that the inertial-conductive subrange should occur for values of the wavenumber $k \gg k_{\text{C-O}}$, where $k_{\text{C-O}}$ is defined in Eq. (5). This translates into $\tau_v \gg \tau_\theta$, i.e., the characteristic time scale of velocity fluctuations is much longer than that of the scalar fluctuations in the inertial-conductive subrange. It thus follows that a reasonable assumption, which will considerably reduce the amount of computational time required for the simulation, is to assume that the velocity field is time independent and its spectral function satisfies Eq. (1). We then need only to time advance the scalar equation.

The code we use is a modification of a code developed by Rogallo.[22] The fluid velocity field is Gaussian and is frozen in an isotropic state, satisfying continuity, and having an energy spectrum of Eq. (1). The temperature field is time advanced (128^3 grid points) according to the unforced scalar equation until the shape of the temperature spectrum becomes independent of its initial state. In a simulation of the inertial-conductive subrange, the temperature field can easily be fully resolved, since the conductivity χ may be chosen as large as desired.

In Fig. 1 the three-dimensional temperature spectral function $G(k)\times k^{17/3}$ vs k is plotted for one realization of Eq. (1). Units are shown in the figure caption. Here $k_{\text{C-O}}$ is the Corrsin–Obukhov wavenumber,

$$k_{\text{C-O}} = (\epsilon/\chi^3)^{1/4}. \qquad (5)$$

We have modified the usual Corrsin–Obukhov units to include the Kolmogorov constant since we have no way of calculating ϵ separately, but only in the combination $\mathrm{Ko}\,\epsilon^{2/3}$. In the units used in Fig. 1, Eq. (4) simplifies to $G(k) = G_0\,\mathrm{Ko}^{-1}k^{-17/3}$.

Figure 1 clearly shows an inertial-conductive subrange of the form of Eq. (4). Averaging over the wavenumbers, $3.5 < k < 8.5$, we find

$$G_0 = 0.33\ \mathrm{Ko} \qquad (6)$$

in excellent agreement with the BHT result. We have also computed $G(k)$ for a few other realizations of Eq. (1), and have found statistical fluctuations of approximately ± 0.02 Ko.

Although the simulation data clearly shows a $k^{-17/3}$ power law subrange, it is yet unclear for which precise range of wavenumbers the result holds. Gibson[13] has suggested that a k^{-3} power law may appear for wavenumbers $k_{\text{C-O}} < k < k_{\text{Ba}}$, where $k_{\text{B}} = \sigma^{-1/4}k_{\text{C-O}}$. Even though we have assumed an inertial range form for the velocity field [Eq. (1)] that, at first glance would imply a vanishing viscosity, in fact, the existence of a cutoff wavenumber k_C in the simulation may be used to infer a viscosity for the fluid. A rough estimate for this effective viscosity may be obtained by

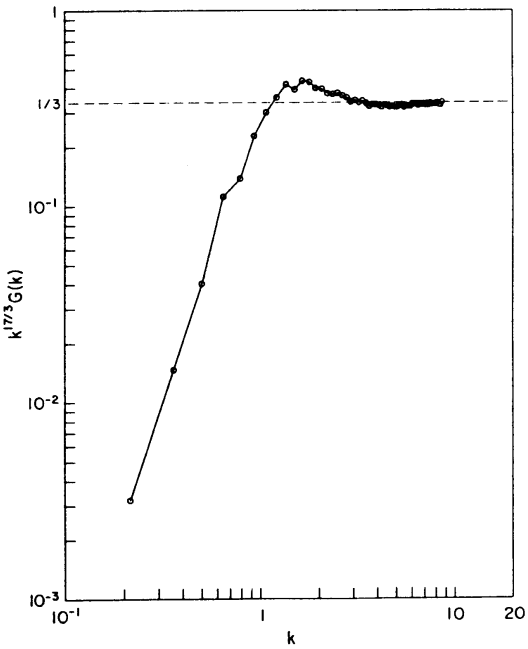

FIG. 1. Three-dimensional temperature spectral function, $G(k)\times k^{17/3}$, convected by a frozen, Gaussian velocity field that satisfies Eq. (1). Here, k is in units of $\mathrm{Ko}^{3/8}\,k_{\text{C-O}}$, while G is in units of $\mathrm{Ko}^{-9/8}(\epsilon\chi^5)^{1/4}(\epsilon_\theta/\epsilon)$.

setting the dissipation wavenumber $k_K = (\epsilon/\nu^3)^{1/4}$ equal to k_C within a factor of order unity. Upon calculation of $k_{C\text{-}O}$ and k_B we find that Gibson's inertial-conductive subrange may be too narrow to allow any additional power law dependence to be observed.

The excellent agreement of Eq. (6) with the original BHT prediction is, in hindsight, perhaps not unexpected. In fact, our main assumptions parallel those underlying the BHT theory; that is, that the velocity field is isotropic with a spectral function satisfying Eq. (1), and that it be considered time independent (frozen) and Gaussian. However, the question remains as to how well real turbulence may be approximated by a frozen, Gaussian velocity field since the velocity field does fluctuate in time and its high order moments (e.g., the skewness and flatness factors) are found to be non-Gaussian at large Reynolds numbers. To study the sensitivity of the temperature spectrum to the frozen, Gaussian field approximation, we have solved the Navier–Stokes equations (64^3 grid points) using a forcing scheme and eddy viscosity subgrid model.[23] First, we evolved the velocity and temperature fields concurrently until an inertial subrange developed. The circles in Fig. 2 show a plot of $k^4G(k)/E(k)$ for this case. Units are shown in the figure caption. We plotted the ratio of $G(k)$ to $E(k)$ in order to smooth out the fluctuations in $G(k)$ caused by deviations in $E(k)$ from an exact $k^{-5/3}$ power law. If $G(k)$ satisfies Eq. (4) and $E(k)$ satisfies Eq. (1), then $k^4G(k)/E(k) = G_0/\text{Ko}$. Second, we took the developed velocity field from the first case, froze it in time, and continued the evolution of the temperature field. The squares in Fig. 2 correspond to this simulation. For comparison, the triangles in Fig. 2 represent a 64^3 simulation using a frozen, Gaussian velocity field.

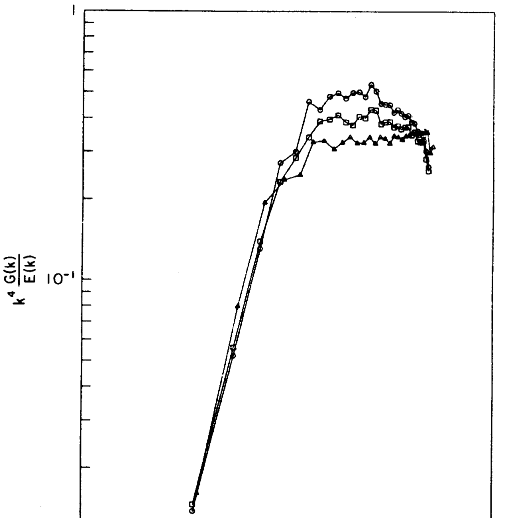

FIG. 2. Three-dimensional temperature spectral function normalized to the three-dimensional energy spectral function, $G(k)/E(k) \times k^4$. Here k is in units of $k_{C\text{-}O}$, while G/E is in units of ϵ_θ/ϵ. The three curves correspond to: △, frozen, Gaussian velocity field; □, frozen, developed velocity field; O, fluctuating, developed velocity field.

It is evident from the three cases shown in Fig. 2 that $G(k)$ best attains a $k^{-17/3}$ spectrum when the velocity field is frozen, the value of G_0 being $G_0 = 0.33 \pm 0.02$ for the case of a Gaussian velocity field and $G_0 = 0.39 \pm 0.03$ for the case of a developed velocity field. The difference in the values of G_0 may be understood in terms of the development of velocity correlations in the latter case as opposed to the former. Since only one realization of the temperature and velocity field is shown in Fig. 2, it is unclear as to the statistical significance in the rise of G_0 from 0.33 to 0.39, although the results presented appear to be typical among the few realizations examined. When the velocity field is allowed to fluctuate in time, the power law behavior of $G(k)$ becomes less clear. However, a simulation performed with a value of the conductivity a factor of 3 greater than that presented in Fig. 2 resulted in the collapse of the temperature spectrum for the fluctuating, developed velocity field (circles in Fig. 2) onto the temperature spectrum for the frozen, developed velocity field (squares in Fig. 2). This behavior is not unexpected since the ratio of the characteristic time scale of the scalar fluctuations, τ_θ, to that of the velocity fluctuations, τ_v, behaves like $\tau_\theta/\tau_v \sim \chi^{-1}$, and thus the frozen field approximation becomes more accurate with increasing conductivity.

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[a] Also with the Department of Physics, Columbia University, New York, New York 10027.

[b] Also with the Department of Physics, City College of New York, New York, New York 10031.

[c] Permanent address: NASA–Ames Research Center, Moffett Field, California 94035.

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