Functions II

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Transformations
Composition
Exponential Functions
Logarithm
Inverse functions
Shifting

• Given a function \( y = f(x) \) and its graph.
• Then the function

\[
y = f(x - b) + d,
\]

where \( b > 0, \ d \) are constants, has the same shape as \( f(x) \) but which is shifted to the right by \( b \) units and shifted up/down by \( d \) units.
• Then the function

\[
y = f(x + b) + d
\]

where \( b > 0, \ d \) are constants, has the same shape as \( f(x) \) but which is shifted to the left by \( b \) units and shifted up/down by \( d \) units.

• A key is to note is that if \( f(x) = y \) reaches the value \( y \) at \( x \), then for what value of a new \( x \) will \( f(x - b) \) to reach the same \( y \)? The same question for \( f(x + b) \).
Transformation figure

Figure: (source: Briggs et al)
General shifting formats

**Vertical and Horizontal Shifts** Suppose $c > 0$. To obtain the graph of

- $y = f(x) + c$, shift the graph of $y = f(x)$ a distance $c$ units upward
- $y = f(x) - c$, shift the graph of $y = f(x)$ a distance $c$ units downward
- $y = f(x - c)$, shift the graph of $y = f(x)$ a distance $c$ units to the right
- $y = f(x + c)$, shift the graph of $y = f(x)$ a distance $c$ units to the left

**Figure:** Table 1 (source: textbook)
Transformations Composition Exponential Functions Logarithm Inverse functions

Shifting figure

Figure: (source: Stewart)
Stretching and reflecting

- The graph of the function

\[ y = c f(a x) \]

has horizontal magnification factor \( a \), and vertical magnification factor \( b \).

**Vertical and Horizontal Stretching and Reflecting** Suppose \( c > 1 \). To obtain the graph of

- \( y = c f(x) \), stretch the graph of \( y = f(x) \) vertically by a factor of \( c \)
- \( y = (1/c) f(x) \), shrink the graph of \( y = f(x) \) vertically by a factor of \( c \)
- \( y = f(cx) \), shrink the graph of \( y = f(x) \) horizontally by a factor of \( c \)
- \( y = f(x/c) \), stretch the graph of \( y = f(x) \) horizontally by a factor of \( c \)
- \( y = -f(x) \), reflect the graph of \( y = f(x) \) about the \( x \)-axis
- \( y = f(-x) \), reflect the graph of \( y = f(x) \) about the \( y \)-axis

**Figure:** (source: Stewart)
Transformation figure I

Figure: 1.38 (source: Briggs, et al)
Transformations
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Stretching and shrinking

$y = cf(x)$
$c > 1$

$y = f(-x)$

$y = \frac{1}{c}f(x)$

$y = -f(x)$

Figure: (source: Stewart)
Stretching trigonometric function

Figure: (source: Stewart)
Stretching and shrinking

\[ y = \cos \frac{1}{2}x \]

\[ y = \cos x \]

\[ y = \cos 2x \]

Figure: (source: Stewart)
Examples of Transformation

- Sketch the graph of $y = (x - 2)^2 - 3$
- Sketch $|2x + 1|$.
- Sketch the graph of $y = 4(x + 3)^2 + 6$
- Sketch the graph according to Exercise 44:
  $$f(x) = \begin{cases} 
    1 - |x| & \text{if } |x| \leq 1; \\
    |x| - 1 & \text{if } |x| > 1.
  \end{cases}$$
(a) $y = -f(x)$, (b) $y = f(x + 2)$, (c) $y = f(x - 2)$, (d) $y = f(2x)$, (e) $y = f(x - 1) + 2$, (f) $y = 2f(x)$.

- Stewart: Ex. 1.3, Q. 3 (p. 74)
Composition

- **Definition** Given two functions $f$ and $g$, their composition $f \circ g$ is a new function, which is defined by

\[(f \circ g)(x) = f(u) = f(g(x))\]

for each $x$ in the domain of $f \circ g$. Let $u = g(x)$ and $y = f(u)$, then $f \circ g$ is understood as

\[y = (f \circ g)(x) = f(g(x)) = f(u), \quad u = g(x),\]

- as shown in

\[x \mapsto u = g(x) \mapsto y = f(u)\]

- with $g$ takes the domain of $g$ (range) into (part of) domain of $f$, and $f$ maps that into (part of) the range of $f$. The two together thus forms a new function $f \circ g$. 
Diagram of composition

Figure: Source: Stewart
Diagram of composition

Figure: 1.8 (source: Briggs, et al)
Examples of composition

- Let \( f(x) = 3x^2 - x \) and \( g(x) = 1/x \).
  1. \( f \circ g \)
  2. \( g \circ f \)
  3. domain and ranges

- (Stewart, Eg. 8) Suppose \( f(x) = x/(x + 1) \), \( g(x) = x^{10} \), and \( h(x) = x + 3 \). Find \( f \circ g \circ h \).

- Recognising composition
  1. \( h(x) = \sqrt{9x - x^2} \)
  2. \( h(x) = \frac{2}{(x^2 - 1)^3} \)

- Given \( f(x) = \sqrt[3]{x} \) and \( g(x) = x^2 - x - 6 \).
  1. \( f \circ g \)
  2. \( g \circ f \)
Composite functions Eg from table

\[
\begin{array}{c|ccccccc}
 x & -1 & 0 & 1 & 2 & 3 & 4 \\
 f(x) & 3 & 1 & 0 & -1 & -3 & -1 \\
 g(x) & -1 & 0 & 2 & 3 & 4 & 5 \\
 h(x) & 0 & -1 & 0 & 3 & 0 & 4 \\
\end{array}
\]

Compute

(a) \( h(g(0)) \),
(b) \( g(f(4)) \),
(c) \( h(h(0)) \),
(d) \( g(h(f(4))) \),
(e) \( f(f(f(1))) \),
(f) \( h(h(h(0))) \),
(g) \( f(g(h(2))) \)

Stewart: Ex. 1.3, Q. 51 (p. 75)
Examples of composition: Gradients

- Given $f(x) = 3x^2 - x$. Find $G(x) = \frac{f(x + h) - f(x)}{h}$

- The above quantity is called the gradient (or slope in less formal language) of $f$ between the two points $x$ and $f(x + h)$. Alternatively, it is the average rate of change of $f$ between $x$ and $x + h$. It is an important quantity as we enter Chapters two and three.

- Let $I = \frac{P}{4\pi r^2}$ measures sound intensity in watts per square meter ($W/m^2$), at a point $r$ meters from a sound source with acoustic power $P = 100 W$. Find the gradient of the secant line through (i) the points $(r_1, I(r_1))$ and $(r_2, I(r_2))$ ($-\frac{P(r_1+r_2)}{4\pi r_1^2 r_2^2} W/m^2$), and (ii) the points $(10, I(10))$ and $(15, I(15))$ ($-1/(36\pi) W/m^2$).
We recognise that real numbers $\mathbb{R}$ are in one-to-one correspondence with decimals on the real line:

![Real Numbers Diagram]

Figure: Stewart

- **Integers**, rationals $m/n$ where $m, n \neq 0$ are integers, **irrationals** (numbers that are not rationals),

$$\pm \sqrt{2}, \pm \sqrt{3}, \sqrt[3]{5}, \ e, \ \pi, \ \log_{10} 5, \ \text{etc}$$

where

$$\sqrt{2} = 1.4142135 \cdots, \ e = 2.718281828 \cdots, \ \pi = 3.141592654 \cdots,$$

have **non-recurrent decimals**. Rationals have recurrent decimals.

- Unlike the rationals, one cannot know the exact locations of an irrational on the real line $\mathbb{R}$. 
Roots

• Theorem

Let $n$ be a positive integer and let $x$ be a fixed positive real number, then there is exactly one positive real number $y$ such that $x = y^n$.

• We denote $y$ by $x^{\frac{1}{n}}$ or $\sqrt[n]{x}$.
• That is, we have $x^{\frac{1}{n}} \cdot x^{\frac{1}{n}} \cdots x^{\frac{1}{n}} = x$.

• So, the square root $2^{\frac{1}{2}}$ means $2^{\frac{1}{2}} \cdot 2^{\frac{1}{2}} = 2$, and the cube root $2^{\frac{1}{3}}$ means $2^{\frac{1}{3}} \cdot 2^{\frac{1}{3}} \cdot 2^{\frac{1}{3}} = 2$, etc.
• Prove that $(x^n)^{\frac{1}{n}} = x$ and hence $(x^n)^{\frac{1}{n}} = (x^{\frac{1}{n}})^n$.
• We define $x^{\frac{m}{n}} = (x^{\frac{1}{n}})^m = (\sqrt[n]{x})^m$.

where $x$ is a given positive real number, and $m$, $n$ are two non-negative integers with $n \neq 0$. 
Roots

- Let $x$ be a positive real number and $n$ a positive integer. We define
  \[ x^{-\frac{1}{n}} = \frac{1}{x^{\frac{1}{n}}}. \]

- We extend $x^{\frac{m}{n}} = (x^{\frac{1}{n}})^m = (\sqrt[n]{x})^m$ to negative rational exponent by
  \[ x^{-\frac{m}{n}} = \frac{1}{x^{\frac{m}{n}}} = \frac{1}{(x^{\frac{1}{n}})^m} = (x^{-\frac{1}{n}})^m, \]
We summarize some direct consequences:

• (i) \( \sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab} \) (for \( (\sqrt[n]{a} \times \sqrt[n]{b})^n = (\sqrt[n]{a})^n(\sqrt[n]{b})^n = ab \));

• (ii) \( \sqrt[n]{a^m} = (\sqrt[n]{a})^m \) (for \( \sqrt[n]{a} \times \cdots \times \sqrt[n]{a^m} = a^m = (\sqrt[n]{a} \times \cdots \times \sqrt[n]{a})^m = (\sqrt[n]{a})^m \times \cdots \times (\sqrt[n]{a})^m \));

• (iii) \( m\sqrt[n]{a} = \sqrt[n]{mn}a \) (for \( (\sqrt[n]{a})^{mn} = a = (\sqrt[n]{a})^n = ((\sqrt[n]{a})^m)^n = (\sqrt[n]{a}^m)^{mn} \));

• (iv) \( \sqrt[n]{a/b} = \sqrt[n]{a} / \sqrt[n]{b} \) (for \( (\sqrt[n]{a/b})^n = a/b = (\sqrt[n]{a})^n / (\sqrt[n]{b})^n = (\sqrt[n]{a} / \sqrt[n]{b})^n \)).

It remains to show, for arbitrary rationals \( p, q \),

(i) \( x^p \cdot x^q = x^{p+q} \);

(ii) \( (x^p)^q = x^{pq} \);

(iii) \( (xy)^p = x^p \cdot x^q \).
We only demonstrate how to establish (i).

(i) Let \( p, q \) be non-negative integers. Then

\[
x^p \cdot x^q = x \cdot \ldots \cdot x \quad \text{with} \quad \underbrace{\underbrace{x \cdot \ldots \cdot x}}_{p \text{ times}} \cdot \underbrace{x \cdot \ldots \cdot x}_{q \text{ times}} = x^{p+q}.
\]

Suppose now that \( p \geq 0, -q > 0 \). Put \( q = -q' \) where \( q' > 0 \). Then

\[
x^p \cdot x^q = x \cdot \ldots \cdot x \quad \text{with} \quad \underbrace{\underbrace{x \cdot \ldots \cdot x}}_{p \text{ times}} \cdot \underbrace{x \cdot \ldots \cdot x}_{q' \times \frac{1}{q' \times \frac{1}{q'}}} = x^{p-q'} \]

We also need to establish the remaining cases when \( -p > 0, q \geq 0 \) and \( -p > 0, -q > 0 \). These are left as your exercises.
Roots

(i) cont. Suppose $p = \frac{m}{n}$, $q = \frac{h}{k}$, where $m$, $n$, $h$, $k$ are integers with $n \neq 0$, $k \neq 0$. In order to prove (i) we need to justify the left-side is equal to the right-side of the relation (i)

- The left-side is

$$x^p \cdot x^q = x^{\frac{m}{n}} \cdot x^{\frac{h}{k}} = x^{\frac{mk}{nk}} \cdot x^{\frac{hn}{kn}}$$

$$= \left(x^{\frac{1}{nk}}\right)^{mk} \cdot \left(x^{\frac{1}{kn}}\right)^{hn}$$

$$= x^{\frac{mk+hn}{nk}}.$$ 

The right-side is simply

$$x^{p+q} = x^{\frac{m}{n} + \frac{h}{k}} = x^{\frac{mk+hn}{nk}}.$$ 

We note that we have used integer power (i) established above. Since both sides are the same, so (i) is completely proved.
Irrational Roots

• Finally, we are face with a more difficult problem, namely to extend the meaning of exponents to irrational numbers, and to justify the rules:
  
  • (i) \( a^x \times a^y = a^{x+y} \);
  • (ii) \((a^x)^y = a^{xy}\);
  • (iii) \(a^x b^x = (ab)^x\)

for arbitrary positive \(a, b\) and arbitrary real numbers \(x, y\).

• We first note that we cannot give a meaning to \(2^{\sqrt{2}-1}\) as fractional exponents above, since \(\sqrt{2} - 1 \approx 0.4142136\) itself is not a fraction. In other words, we do not know what it means except that we have a notation. However, it is important to give it a meaning since the function \(2^x\) will have many occasion when the index \(x\) becomes an irrational as \(x\) changes from 0 to 1 say.
Irrational Roots

- An elementary way to understand $2\sqrt{2} - 1$ is to consider rational approximation to $\sqrt{2} - 1$. Since $\sqrt{2} - 1 = 0.4142135 \cdots$, then

\[
\frac{5}{10}, \frac{42}{100}, \frac{415}{1,000}, \frac{4144}{1,0000}, \cdots
\]

are all larger than $\sqrt{2} - 1$, but are nonetheless decreasing rationals and approaches $\sqrt{2} - 1$ from above. We also can find a corresponding sequence of rationals such that each of which is smaller than $\sqrt{2} - 1$ and approaches it from below. In summary, we have

\[
0.4 < 0.41 < 0.414 < 0.412 < \cdots < \sqrt{2} - 1 < \cdots < 0.4143 < 0.415 < 0.42 < 0.5.
\]
Irrational Roots

- In fact any irrational number can be approximated by two such sequences of rational numbers as $\sqrt{2} - 1$ above. We may even define irrationals as all those real numbers that could be approximated by rationals this way. Hence we may define $2^{\sqrt{2}-1}$ to be the unique real number so that

$$
2^{0.4} < 2^{0.41} < 2^{0.414} < 2^{0.4142} < \ldots < 2^{\sqrt{2}-1} < \ldots < 2^{0.4143} < 2^{0.415} < 2^{0.42} < 2^{0.5}.
$$

- This is because 2 to a rational exponent has already been defined and so has a meaning. The other irrational exponents can be defined similarly. It is not difficult to see that the three exponents rules (i), (ii) and (iii) also hold by the above reasoning (of course we omit vigorous reasoning behind these rules since they are well beyond the scope of this course).
Exponential Functions

- The function

\[ y = f(x) = 2^x \]

is called \textit{binary exponential function with base 2}.

- Let \( b \neq 1 \) be an arbitrary positive number. Then function

\[ y = f(x) = b^x \]

is called \textit{exponential function with base} \( b \).

- The exponential function grows very \textit{fast without bound} if \( b > 1 \) and \textit{tends} to zero fast.

- Amongst all exponential functions, the \textit{natural exponential function}

\[ y = f(x) = e^x \]

where \( 2 < e \approx 2.71828 < 3 \) is amongst the most important.

- The simpler one is the one with \textit{common base 10}

\[ y = f(x) = 10^x \]
Logarithm

- \( b^x \times b^y = b^{x+y} \), \( b^x \div b^y = b^{x-y} \)
- \( (b^x)^y = b^{x\cdot y} \).

These observations, about turning multiplication or division into addition or subtraction of the exponents respectively, may seem elementary, but it actually becomes so important in all kinds of applications, some quite unexpected.

- The application of these rules were known since Babylonians’ time in 2000-1600 BC. But it was the Scottish John Napier (1550-1617) who wrote the book entitled *Mirifici Logarithmorum Canonis Descriptio* (Description of the Wonderful Rule of Logarithms) in 1614 that popularized their use (source from Wiki)

- First called artificial number, then ”logarithm” meaning from Latin “proportional-arithmetic

John Napier

Figure: (1550-1617) Source from Wiki
Tables published by Cambridge

Figure: (Source from Wiki)
A page in Log tables

Figure: (Source from Wiki)
Notation of Logarithm

- Let us recall that after agreeing to a base $b$, we make use of

$$b^x \times b^y = b^{x+y}, \quad b^x \div b^y = b^{x-y}.$$  

- To simplify the writing, we need notation that shows, in the case of multiplication, only the $x$, $y$ and $x+y$ and to de-emphasis the base $b$.

- Without loss of generality, we may assume to multiply two positive numbers $X$, $Y$. We first need to turn them into exponents of $b$. That is, suppose we can find positive numbers $x$, $y$ such that

$$b^x = X, \quad b^y = Y.$$  

- This works if there is a unique $x$ and $y$ that correspond to $X$, $Y$ respectively. We want to work with $x$ and $y$ only.

- Denote $x = \log_b X$ and $y = \log_b Y$. 

**Rules of Logarithm**

- **Definition** Given any \( X > 0 \), we define the exponent \( x \) for which \( b^x = X \) to be the logarithm of \( X \) with respect to the base \( b \). The \( x \) is commonly denoted by \( x = \log_b X \). That is,

\[
X = b^x = b^{\log_b X}. \quad \text{(so } \log_b 1 = 0)\]

- We write \( X \ Y = X \ Y \) in terms of the base \( b \) in two ways

\[
b^{\log_b X} \times b^{\log_b Y} = b^{\log_b XY}.
\]

But \( b^x \times b^y = b^{x+y} \). So

\[
\log_b X + \log_b Y = \log_b XY
\]

- Then one can deduce from \( b^x \div b^y = b^{x-y} \) the relation

\[
\log_b X - \log_b Y = \log_b \frac{X}{Y}
\]
Rules of Logarithm II

- By the definition of logarithm (that is $X = b^x = b^{\log_b X}$),

$$a^m = b^{\log_b(a^m)} = b^{(\log_b a + \cdots + \log_b a)} = b^{m \log_b a},$$

so

$$\log_b a^m = m \log_b a$$

holds when $m$ is a positive integer. If $m = -n$ where $n$ is a positive integer, the logarithm is still valid:

$$a^m = a^{-n} = \frac{1}{a^n} = \frac{1}{b^{\log_b(a^n)}} = \frac{1}{b^{n \log_b a}} = b^{-n \log_b a} = b^{m \log_b a}.$$

- If $a^x$ where $x = \frac{r}{s}$ is a fraction, then we interpret

$$a^x = a^{\frac{r}{s}} = (a^{\frac{1}{s}})^r.$$

Now proceed with the previous argument.
Rules of Logarithm III

- How do these different logarithms for the same number $X$ relate to each other?
- Suppose $X$ is a given number, and
  \[ b^{\log_b X} = X = c^{\log_c X} \]
  are the two different logarithms with respect to base $b$ and $c$ respectively.
- Taking $\log_b$ on both sides yields
  \[ (\log_b X)(\log_b b) = (\log_c X)(\log_b c) \]
  \[ = 1 \]
  That is, $\log_b X / \log_c X = \log_b c$.
- Similarly, Taking $\log_c$ on both sides yields
  \[ (\log_b X)(\log_c b) = (\log_c X)(\log_c c) \]
  \[ = 1 \]
  That is, $\log_b X / \log_c X = 1 / \log_c b$.
- So $\log_b c = 1 / \log_c b$. 
Rules of Logarithm IV

- Let us consider again $X$ is a given number, and

$$b^{\log_b X} = X = c^{\log_c X}$$

- Then

$$X = b^{\log_b X} = (c^{\log_c b})^{\log_b X} = c^{(\log_c b)(\log_b X)}$$

and

$$X = c^{\log_c X} = (b^{\log_b c})^{\log_c X} = b^{(\log_b c)(\log_c X)}$$

Thus

$$\log_b X = (\log_b c)(\log_c X) = (\log_b c)(\log_c b)(\log_b X)$$

Hence

$$\log_b c = \frac{1}{\log_c b}.$$
• Consider $\log_a b$ and change its base to $c$, say.
• That is, if $\log_a b = x$. Then $a^x = b$.
• So
  $$c^{\log_c b} = b = (c^{\log_c a})^x = c^x \log_c a.$$  
But then
  $$\log_a b = x = \frac{\log_c b}{\log_c a}$$
as required.
Logarithm examples

• Solve the following equations (Ex (Briggs, et al) 1.3: Q. 41-46):
  - \( \log_{10} x = 3 \)
  - \( \log_5 x = -1 \)
  - \( \log_8 x = 1/3 \)
  - \( \log_b 125 = 3 \)

• Solve the following equations (Ex (Briggs, et al) 1.3: Q. 53-56):
  - \( 7^x = 21 \)
  - \( 3^{3x-4} = 15 \)

• Convert the following logarithms into common base 10:
  - \( \log_2 15 \)
  - \( \log_3 30 \)

• Express \( 2^x \) with base \( e \)

• Express \( 3^{\sin x} \) with base \( e \)
Inverse functions

**Definition** Let $f$ be a function defined on its domain $D$. Then a function $f^{-1}$ is called an inverse of $f$ if

$$(f^{-1} \circ f)(x) = x,$$  
for all $x$ in $D$.

That is, $x = f^{-1}(y)$ whenever $y = f(x)$.

**Remark 1** It follows that the domain of $f^{-1}$ is on the range of $f$.

**Remark 2** There is no guarantee that every function has an inverse.

**Remark 3** If $f$ has two inverse functions, then the two inverse functions must be identically the same.
Indication of inverse functions A

Figure: Briggs, et al: 1.49a
Indication of inverse functions b

Figure: Briggs, et al: 1.49b
Domain-Range

domain of $f^{-1} = \text{range of } f$

range of $f^{-1} = \text{domain of } f$

Figure: Stewart: page 60
Examples of inverse function

• Let \( y = f(x) = ax + b \) where \( a, b \) are constants. Then

\[
x = \frac{y - b}{a}
\]

is the inverse function of \( f \). That is,

\[
x = f^{-1}(y) = \frac{1}{a}(y - b), \quad x = (f^{-1} \circ f)(x).
\]

• \( y = f(x) = 2x + 6 \).

\[
x = f^{-1}(y) = \frac{1}{2}(y - 6).
\]

• In fact, a criterion of a given \( f \) has an inverse is that it is an injective \((one-one)\) mapping.

• In terms of the graph of \( y = f(x) \), the graph of \( f \) is either increasing or decreasing against the \( x - axis \).
Inverse linear functions

\[ f(x) = 2x + 6 \]

\[ f^{-1}(x) = \frac{x}{2} - 3 \]

\[ y = x \]

**Figure:** (Publisher) 1.52
Graphing of inverse functions

- (Briggs, et al: p. 31) \( y = f(x) = x^2 - 1 \).
- Choose inverse as \( x = +\sqrt{y + 1} \).
- We really consider \( x \) as a function of \( y \) now.
- Convention is that we use \( x \) for the independent variable.
- Interchange the \( x \) and \( y \).

\[
y = \sqrt{x + 1}.
\]

- A practical graphical procedure to find the graph of the inverse \( y = f^{-1}(x) \) is to rotate the graph of \( y = f(x) \) along the straight line \( x = y \) by 180 degrees in our three dimensional space.
- Can apply this for \( x = f^{-1}(y) = \frac{1}{2}(y - 6) \).
Inverse a quadratic function example

\[ f^{-1}(x) = x^2 + 1 \quad (x \geq 0) \]

\[ f(x) = \sqrt{x - 1} \]

Figure: (Publisher) 1.53
Inverse of quadratic function

- Let $y = f(x) = ax^2 + bx + c$, where $a > 0$.
- We apply the method of completing the square:

\[
y = f(x) = ax^2 + bx + c \quad (a \neq 0)
\]

\[
= a \left[ x^2 + \frac{b}{a}x + \frac{c}{a} \right]
\]

\[
= a \left[ x^2 + 2 \left( \frac{b}{2a} \right) x + \left( \frac{b}{2a} \right)^2 + \frac{c}{a} - \left( \frac{b}{2a} \right)^2 \right]
\]

\[
= a \left[ \left( x + \frac{b}{2a} \right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} \right]
\]

\[
= a \left[ \left( x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a^2} \right]
\]

\[
= a \left( x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a}.
\]
Inverse of quadratic function

• That is, we have

\[ x = -\frac{b}{2a} \pm \sqrt{\frac{1}{a} \left( y + \frac{b^2 - 4ac}{4a} \right)}. \]

Recall that the solution of \( x \) when \( y = 0 \) are possible only when \( b^2 - 4ac \geq 0 \).

• Choosing the “+” branch and switching the roles of \( x \) and \( y \) yields

\[ y = f^{-1}(x) = -\frac{b}{2a} + \sqrt{\frac{1}{a} \left( x + \frac{b^2 - 4ac}{4a} \right)}. \]

• Explicit inverse functions are actually difficult to find.
Exponential functions graphs

Figure: Stewart: S 1.4, figure 13
Exponential and logarithmic graphs

**Figure:** Stewart: S 1.6, figure 11
Logarithmic function graphs

Figure: Stewart: S 1.6, figure 12
Logarithm as inverse function

- If we view $y = f(x) = b^x$ as a given function, then its inverse is given by $y = f^{-1}(x) = \log_b x$ since we can check
  $$(f^{-1} \circ f)(x) = \log_b(b^x) = x$$
  by the definition of logarithm.

- In fact, even
  $$(f \circ f^{-1})(x) = b^{\log_b x} = x$$
  holds trivially.

- The graph of $\log_b x$ is obtained from rotating $y = b^x$ along the line $x = y$ by 180 degrees.
Solving logarithm as inverse function

• (Briggs, et al §1.3 Example 5) One thousand grams of a particular radioactive substance decays according to the function \( m(t) = 1000 e^{-t/850} \), measured in grams, where the time \( t \geq 0 \) is measured in years. When does the mass of the substance reach the safe level of 1 gram?

We want to know for which time \( t \) that \( m(t) = 1 \). That is, we want to solve

\[
\ln e^{-t/850} = \ln \frac{1}{1000}.
\]

Hence

\[
-\frac{t}{850} \approx -6.908,
\]

or \( t \approx (-850)(-6.908) = 5871.8 \) years.
## Slow-growing logarithmic function

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>5</th>
<th>10</th>
<th>50</th>
<th>100</th>
<th>500</th>
<th>1000</th>
<th>10,000</th>
<th>100,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ln x$</td>
<td>0</td>
<td>0.69</td>
<td>1.61</td>
<td>2.30</td>
<td>3.91</td>
<td>4.6</td>
<td>6.2</td>
<td>6.9</td>
<td>9.2</td>
<td>11.5</td>
</tr>
<tr>
<td>$\sqrt{x}$</td>
<td>1</td>
<td>1.41</td>
<td>2.24</td>
<td>3.16</td>
<td>7.07</td>
<td>10.0</td>
<td>22.4</td>
<td>31.6</td>
<td>100</td>
<td>316</td>
</tr>
<tr>
<td>$\frac{\ln x}{\sqrt{x}}$</td>
<td>0</td>
<td>0.49</td>
<td>0.72</td>
<td>0.73</td>
<td>0.55</td>
<td>0.46</td>
<td>0.28</td>
<td>0.22</td>
<td>0.09</td>
<td>0.04</td>
</tr>
</tbody>
</table>

**Figure:** Stewart: page 66
Slow-growing logarithmic function

\[ y = \ln x \]

\[ y = \sqrt{x} \]

Figure: Stewart: Figure 15
Slow-growing logarithmic function

\[ y = \sqrt{x} \]

\[ y = \ln x \]

Figure: Stewart: Figure 16