2.7.3 Trigonometric Substitution

Let us recall the trigonometric identities.

Suppose the circle has radius 1, then the Pythagorean theorem becomes

\[
\sin^2 \theta + \cos^2 \theta = 1 \\
\tan^2 \theta + 1 = \sec^2 \theta \\
1 + \cot^2 \theta = \csc^2 \theta \\
\sqrt{1-x^2} = \sqrt{1-\cos^2 \theta} = \sin \theta \\
\sqrt{1-x^2} = \sqrt{1-\sin^2 \theta} = \cos \theta \\
\sqrt{1+x^2} = \sqrt{1+\tan^2 \theta} = \sec \theta \\
\sqrt{x^2-1} = \sqrt{\sec^2 \theta - 1} = \tan \theta = \cot \theta
\]

All these identities can help with integrations. If it is not a unit circle, but a circle with \(a\), then we scale all factors by \(a\).
We recall that since \((\tan \theta)' = \sec^2 \theta\) and \((\cot \theta)' = -\csc^2 \theta\)

So

\[
\int \sec^2 \theta \, d\theta = \tan \theta + C, \quad \int \csc^2 \theta \, d\theta = -\cot \theta + C
\]

**Example** Evaluate \(\int \frac{\sqrt{9-x^2}}{x^2} \, dx\)

\[
\int \frac{\sqrt{9-x^2}}{x^2} \, dx = 3 \int \frac{1-\left(\frac{x}{3}\right)^2}{\frac{x^2}{9}} \, dx = 3 \int \frac{1-u^2}{9u^2} \, du
\]

\[
u = \sin \theta, \quad \frac{du}{d\theta} = \cos \theta \Rightarrow \int \frac{1-\sin^2 \theta}{\sin^2 \theta} \cos \theta \, d\theta = \int \csc^2 \theta \, d\theta = -\cot \theta + C
\]

\[
= \int \csc^2 \theta - 1 \, d\theta = -\cot \theta - \theta + C
\]

To invert back to \(x\), we need to know what \(\cot \theta\) means in terms of \(x\). This is done via the side length of the right-angled triangle that we made the substitution \(x = 3 \sin \theta\).

Hence we need to restrict the \(\theta\) to be \(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\).

![Diagram of a right-angled triangle with \(x/3\) as the hypotenuse, 1 as the opposite side, and \(\theta\) as the angle between the hypotenuse and the adjacent side.]

\[
\cot \theta = \frac{1-(\frac{x}{3})^2}{\frac{x^2}{9}} = \frac{\sqrt{9-x^2}}{x}
\]

Thus we deduce

\[
\int \frac{\sqrt{9-x^2}}{x^2} \, dx = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1} \frac{x}{3} + C
\]
Remark. The range of \( \theta \) for which the substitution \( x = 3 \sin \theta \) is valid is \(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\). The right-angled triangle constructed above restricts \( 0 < \theta < \frac{\pi}{2} \). It can be shown that the answer remains valid for \(-\frac{\pi}{2} < \theta < 0\).

Example. Find the area of the ellipse defined by

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]

It is sufficient to compute the area lying in the first quadrant.

\[
y = \pm b \sqrt{1 - \frac{x^2}{a^2}} = \pm \frac{b}{a} \sqrt{a^2 - x^2} \quad (0 \leq x \leq a)
\]

\[
\left( \text{Area in the 1st quadrant} \right) = \int_0^a y \, dx = \int_0^a \frac{b}{a} \sqrt{a^2 - x^2} \, dx
\]

\[
\frac{\sin \theta}{dx} = a \cos \theta \, d\theta + \frac{a \cos \theta \, d\theta}{\cos \theta} = ab \int_0^{\pi/2} \cos^3 \theta \, d\theta
\]

\[
= ab \int_0^{\pi/2} (1 + \cos 2\theta) \frac{1}{2} \, d\theta = \frac{ab}{2} \left( \theta + \frac{1}{4} \sin 2\theta \right)_0^{\pi/2} = \frac{ab \pi}{4}
\]

Hence the total area of the ellipse is \( \pi ab \).
Example \( \int \frac{dx}{x^2 + 4} \)

Let \( x = 2 \tan \theta \). Then \( dx = 2 \sec^2 \theta \, d\theta \)

\[
\int \frac{dx}{x^2 + 4} = \int \frac{2 \sec^2 \theta}{4 \tan \theta \cdot 2 \sec \theta} \, d\theta = \frac{1}{4} \int \frac{\sec \theta}{\tan \theta} \, d\theta
\]

\[
= \frac{1}{4} \int \frac{1}{\cos \theta} \cdot \frac{\cos^2 \theta}{\sin^2 \theta} \, d\theta = \frac{1}{4} \int \frac{\cos \theta}{\sin \theta} \, d\theta
\]

\[
= \frac{1}{4} \frac{-1}{\sin \theta} + C = -\frac{1}{4 \sin \theta} + C
\]

\[
\sin \theta = \frac{x}{\sqrt{x^2 + 4}}
\]

Here

\[
\int \frac{dx}{x^2 + 4} = -\frac{1}{4} \frac{x^2 + 4}{4} + C
\]

Example \( \int \frac{dx}{\sqrt{x^2 - a^2}} \) (a > 0)

Let \( x = a \sec \theta \). Then \( dx = a \sec \theta \tan \theta \, d\theta \) \((0 < \theta < \frac{\pi}{2}) \) or \( \frac{\pi}{2} < \theta < \frac{3\pi}{2} \) because of the problem of the definition of \( \sec^2 x \), see below)

Thus
\[ \int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{a \sec \theta \tan \theta}{a \tan \theta} \, d\theta = \int \sec \theta \, d\theta \]

\[ = \ln |\sec \theta + \tan \theta| + C \]

We note that
\[ \sqrt{x^2 - a^2} = \sqrt{a^2(\sec^2 \theta - 1)} = \sqrt{a^2 \tan^2 \theta} = a \tan \theta \]

\[ = a \tan \theta \]

because \( \tan \theta > 0 \) for \( 0 < \theta < \frac{\pi}{2} \) and \( \pi < \theta < \frac{3\pi}{2} \).

Since \( \sec \theta = \frac{x}{a} \), so \( \tan \theta = \frac{\sqrt{x^2 - a^2}}{a} \)

Therefore
\[ \int \frac{dx}{\sqrt{x^2 - a^2}} = \ln \left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right| + C \]

Example \( \int \frac{x}{\sqrt{3 - 2x - x^2}} \, dx \)

We apply the method of completing the square to

\[ 3 - 2x - x^2 = -(x^2 + 2x - 3) = -((x+1)^2 - 4) = 2^2 - (x+1)^2 \]
Let \( u = (x+1) \), \( du = dx \). Thus

\[
\int \frac{x \, dx}{\sqrt{3 - 2x - x^2}} = \int \frac{(u-1) \, du}{\sqrt{4 - u^2}}
\]

\( u = 2 \sin \theta \)

\[
\frac{du}{d\theta} = 2 \cos \theta \, d\theta
\]

and

\[
2 \cos \theta = \sqrt{4 - u^2}
\]

Thus

\[
\int \frac{(2 \sin \theta - 1) \, d\theta}{\sqrt{4 - u^2}} = -2 \cos \theta - \theta + C
\]

\[
= -\sqrt{4 - u^2} - \sin^{-1}\left(\frac{u}{2}\right) + C
\]

\[
= -\sqrt{3 - 2x - x^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C
\]