7.4 Integration via Partial Fractions

Let

\[ f(x) = \frac{P(x)}{Q(x)} \]

where \( P(x) \) and \( Q(x) \) be polynomials. We want to compute

\[ \int f(x) \, dx = \int \frac{P(x)}{Q(x)} \, dx, \]

where we may assume \( \deg P(x) < \deg Q(x) \). For if \( \deg P(x) \geq \deg Q(x) \), the division algorithm allows us to write

\[ f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)} \]

where \( S(x) \) is a polynomial, and \( \deg R(x) < \deg Q(x) \).

Example

\[ \int \frac{x^3 + x}{x - 1} \, dx. \]

Since

\[ \frac{x^3 + x}{x - 1} \, dx = x^2 + x + 2 + \frac{2}{x - 1}, \]

So

\[ \int \frac{x^3 + x}{x - 1} \, dx = \int x^2 + x + 2 + \frac{2}{x - 1} \, dx \]

\[ = \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2 \ln |x - 1| + C \]

Q: But how do we deal with general \( \frac{P(x)}{Q(x)} \)?

Suppose we want to compute

\[ \int \frac{x^3 + x + 1}{x(x-1)(x^2+1)(x+1)^3} \, dx \]
The method of Partial fraction allows us to write

\[
\begin{array}{c|c}
\text{partial fractions} & \frac{x^3 + x^2 + 1}{x(x-1)(x^2+x+1)(x^2+1)^3} \\
\hline
\text{Result} & \frac{x^3 + x^2 + 1}{(x - 1)x(x^2 + 1)^2(x^2 + x + 1)} = \\
& -\frac{x - 1}{x^2 + x + 1} + \frac{15x - 1}{8(x^2 + 1)} + \frac{3(x + 1)}{4(x^2 + 1)^2} + \frac{1 - x}{2(x^2 + 1)^3} + \frac{1}{8(x - 1)} - \frac{1}{x} \\
\end{array}
\]

Mathematica

Now it more likely that we can integrate the original rational function by integration of each decomposed terms.

But the general theory requires the use of complex function theory (from the Math Dept.) which lies outside the scope of this course.

We contend to describe the algorithm of partial fractions.

Partial fractions method is not just a useful tool for integration, a quick reach by typing the keyword "Partial Fractions" returns the following respond:
A p-adic algorithm for univariate partial fractions

PS Wang - Proceedings of the fourth ACM symposium on ..., 1981 - dl.acm.org

Abstract: Partial fractions is an important algebraic operation with many applications in applied mathematics, physics and engineering. It is also an important operation in any computer symbolic and algebraic system. Among other things, it is used in the integration ...

Incomplete partial fractions for parallel evaluation of rational matrix functions

D Calvetti, E Gallopoulos, L Reichel - Journal of computational and applied ..., 1995 - Elsevier

Frequently, one needs to evaluate expressions of the form \( p(A) \) \( q(B) \), where \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{m \times m} \), and \( p \) and \( q \) are polynomials with degree \( q \) \( m \), such that no zero of \( p \) is an eigenvalue of \( A \). Algorithms based on the partial fraction representation of \( q \) when ...

Rational approximation of frequency domain responses by vector fitting

B Gustavsen, A Semlyen - IEEE Transactions on ..., 1999 - ieeexplore.ieee.org

... Such responses have been approximated by fitting partial fractions to the data in an optimization procedure, with precalculated poles [4]. An attempt at formulating a general fitting methodology was introduced in [5]. This method-vector fitting-was based on doing the ...

Partial fractions expansion: a review of computational methodology and efficiency

JF Mahoney, BD Sivazlian - Journal of Computational and Applied ..., 1983 - Elsevier

Abstract: Nine methods for expressing a proper rational function in terms of partial fractions are presented for the case where the denominator polynomial has been reduced to linear factors. Only those methods which are amenable to computation algorithms are ...

Partial fractions and bilateral summations

W Chu - Journal of Mathematical Physics, 1994 - scitation.aip.org

Bilateral extensions of the hypergeometric formulas due to Minton (1970), Karlsson (1971), and Gasper (1981) are established through the decomposition of products of gamma functions into partial fractions. Some exceptional cases are demonstrated and a ...

A use of complex probabilities in the theory of stochastic processes

DR Cox - Mathematical Proceedings of the Cambridge ..., 1955 - Cambridge Univ Press

... Let \( X \) be in \( \mathcal{X} \) and be the ratio of a polynomial of degree at most \( k \) to a polynomial of degree \( k \). Then if the zeros of the denominator are at \( -A^i (i = 1, \ldots, k) \), we can expand \( f(x) \) in partial fractions in many ways, of which we still consider two, viz. \( f(x) = \frac{g(x)}{h(x)} + \sum_{i=1}^{k} \frac{a_i}{x - A^i} \) ...

Partial fractions, and four classical theorems of number theory
We distinguish a number of different cases based on the level of complexity.

**CASE I** The denominator \( Q(x) \) is a product of distinct linear factors.

This means that we can write

\[
Q(x) = (a_1x + b_1)(a_2x + b_2) \cdots (a_kx + b_k)
\]

where no factor is repeated (and no factor is a constant multiple of another). In this case the partial fraction theorem states that there exist constants \( A_1, A_2, \ldots, A_k \) such that

\[
\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}
\]

Stewart

\[
\text{Example} \quad \text{Integrate} \quad \int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} \, dx
\]

Consider

\[
2x^3 + 3x^2 - 2x = x(2x-1)(x+2).
\]

According to the **Case I**, we have

\[
\frac{x^2 + 2x - 1}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}
\]

\[
= \frac{A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)}{x(2x-1)(x+2)}
\]

\[
= \frac{A(2x^2 + 3x - 2) + B(x^2 + 2x) + C(2x^2 - x)}{x(2x-1)(x+2)}
\]

\[
= \frac{(2A + B + 2C)x^2 + (3A + 2B - C)x - 2A}{x(2x-1)(x+2)}
\]
Comparison the coefficients on both sides yield

\[-2A = -1 \text{ so that } A = \frac{1}{2}. \text{ We also have }\]

\[1 = 2A + B + 2C = 1 + B + 2C\]

and

\[2 = 3A + 2B - C = \frac{3}{2} + 2B - C\]

Solving the last two equations yield \(B = \frac{1}{5}\). Thus

\[B = \frac{1}{10}. \text{ Hence }\]

\[
\frac{1}{x(2x-1)(x+2)} = \frac{1}{2x} + \frac{1}{5(2x-1)} - \frac{1}{10(x+2)}
\]

Hence

\[
\int \frac{x^2 + 2x - 1}{x(2x-1)(x+2)} \, dx = \int \frac{1}{2x} + \frac{1}{5(2x-1)} - \frac{1}{10(x+2)} \, dx
\]

\[= \frac{1}{2} \ln |x| + \frac{1}{10} \ln |2x-1| - \frac{1}{10} \ln |x+2| + C
\]

Example. \(\int \frac{dx}{x^2 - a^2} (a > 0)\)

We can compute

\[
\frac{1}{x^2 - a^2} = \frac{A}{x-a} + \frac{B}{x+a} = \frac{A(x+a) + B(x-a)}{x^2 - a^2} = \frac{(A+B)x + a(A-B)}{x^2 - a^2}
\]

Hence \(A + B = 0\) and \(a(A - B) = 1\). We deduce \(A = \frac{1}{2a}, \ B = \frac{1}{2a}.\)

Then

\[
\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \int \frac{1}{x-a} - \frac{1}{x+a} \, dx
\]

\[= \frac{1}{2a} \left( \ln |x-a| - \ln |x+a| \right) + C = \frac{1}{2a} \ln \frac{|x-a|}{|x+a|} + C.
\]
CASE II. \( Q(x) \) is a product of linear factors, some of which are repeated.
Suppose the first linear factor \((a_1x + b_1)\) is repeated \( r \) times; that is, \((a_1x + b_1)^r\) occurs in the factorization of \( Q(x) \). Then instead of the single term \( A_1/(a_1x + b_1) \) in Equation 2,

\[
\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r}
\]

Example \( \int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 + x + 1} \, dx \)

\[
= \int \left( x + 1 + \frac{4x}{x^3 - x^2 - x + 1} \right) \, dx = \int x + 1 + \frac{4x}{(x-1)^3} \, dx - \frac{1}{x+1} \, dx
\]

\[
= \frac{x^2}{2} + x + \ln |x-1| - \frac{2}{x-1} - \frac{1}{x+1} + C = \frac{x^2}{2} + x - \frac{2}{x-1} + \ln \frac{x-1}{x+1} + C
\]

where according to the above computational rule for repeated factors,

\[
\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} = \frac{A(x-1)(x+1) + B(x+1) + C(x-1)^2}{(x-1)^2(x+1)}
\]

\[
= (A+C)x^2 + (B-2C)x + (A-B+C)
\]

Hence \( A+C = 0 \), \( B-2C = 4 \) and \( -A+B+C = 0 \). Thus \( A = -C \) and \( B+2C = 0 \). We deduce \( 2B = 4 \), \( B = 2 \). Thus \( C = -1 \) and \( A = 1 \).
CASE III  $Q(x)$ contains irreducible quadratic factors, none of which is repeated.

If $Q(x)$ has the factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then, in addition to the partial fractions in Equations 2 and 7, the expression for $R(x)/Q(x)$ will have a term of the form

$$\frac{Ax + B}{ax^2 + bx + c}$$

where $A$ and $B$ are constants to be determined.

That is, the factor $an^2 + bn + c$ cannot be factored without the use of complex numbers. Let us recall the formula

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$

Example

$$\int \frac{2x^2 - x + 4}{x^2 + 4} \, dx$$

Hence

$$\frac{2x^2 - x + 4}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4} = \frac{A(x^2 + 4) + x(Bx + C)}{x(x^2 + 4)}$$

Thus

$$A + B = 2, \ C = -1, \ 4A = 4.$$ 

Hence $A = 1, \ B = 1$ 

Thus

$$\int \frac{2x^2 - x + 4}{x^2 + 4} \, dx = \int \frac{1}{x} + \frac{x - 1}{x^2 + 4} \, dx =$$

$$= \int \frac{1}{x} \, dx + \int \frac{x}{x^2 + 4} \, dx - \int \frac{1}{x^2 + 4} \, dx$$

$$= \ln|x| - \frac{1}{2} \ln(x^2 + 4) - \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + C$$
Example \[ \int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3} \, dx \]

We first note that

\[
\frac{4x^2 - 3x + 2}{4x^2 - 4x + 3} = 1 + \frac{x-1}{4x^2 - 4x + 3} = 1 + \frac{x-1}{(2x-1)^2 + 2}
\]

So

\[
\int \frac{4x^2 - 3x + 2}{4x^2 - 4x + 3} \, dx = \int 1 + \frac{x-1}{(2x-1)^2 + 2} \, dx
\]

\[
= \int dx + \int \frac{x}{(2x-1)^2 + 2} \, dx - \int \frac{1}{(2x-1)^2 + 2} \, dx
\]

Let \( u = 2x - 1 \) then \( du = 2 \, dx \)

\[
x + \frac{1}{2} \ln \left( \frac{u+1}{u^2 + 2} \right) - \frac{1}{4} \ln \left( \frac{u+1}{u^2 + 2} \right) + C
\]

\[
= x + \frac{1}{8} \ln \left( (2x-1)^2 + 2 \right) - \frac{1}{4} \ln \tan^{-1} \left( \frac{2x-1}{\sqrt{2}} \right) + C
\]

\[
= x + \frac{1}{8} \ln \left( 4x^2 - 4x + 3 \right) - \frac{1}{4\sqrt{2}} \ln \tan^{-1} \left( \frac{2x-1}{\sqrt{2}} \right) + C
\]

CASE IV \( Q(x) \) contains a repeated irreducible quadratic factor.

If \( Q(x) \) has the factor \((ax^2 + bx + c)'\), where \( b^2 - 4ac < 0 \), then instead of the single partial fraction \([9]\), the sum

\[
\sum_{i=1}^{r} \frac{A_i x + B_i}{(ax^2 + bx + c)^i}
\]

\[\text{Stewart}\]
Let us recall that

\[
\frac{x^3 + x^2 + 1}{x(x-1)(x^2+1)^3(x^2+x+1)} = \\
\frac{-x-1}{x^2 + x + 1} + \frac{15x-1}{8(x^2+1)} + \frac{3(x+1)}{4(x^2+1)^2} + \frac{1-x}{2(x^2+1)^3} + \frac{1}{8(x-1)} - \frac{1}{x}
\]

Mathematica

That we saw earlier illustrate this repeated quadratic factors.

Example \(\int \frac{1-x+2x^2-x^2}{x(x^2+1)^2} \, dx\)

We consider

\[
\frac{1-x+2x^2-x^2}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}
\]

\[
A(x^2+1)^2 + (Bx+C)x(x^2+1) + x(Dx+E)
\]

\[
= \frac{\chi(x^2+1)^2}{x}\n\]

\[
= \chi(x^2+1)^2
\]

\[
= \frac{(A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A}{x(x^2+1)^2}
\]

Hence \(A=1\), \(A+B=0\) so that \(B=-1\). But \(C=-1\) and \(2A+B+D=2\) implies that \(D=1\). Similarly \(C+E=-1\) implies \(E=0\).
\[
\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2} \, dx = \int \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2} \, dx
\]

\[
= \int \frac{dx}{x} - \int \frac{x+1}{x^2+1} \, dx + \int \frac{x}{(x^2+1)^2} \, dx
\]

\[
= \ln|x| - \frac{1}{2} \ln(x^2+1) - \arctan x - \frac{1}{2} \frac{1}{(x^2+1)} + C
\]

Remarks 1: Observation is still the key for the best strategy for

\[
\int \frac{x^3+1}{x(x^2+3)} \, dx
\]

which can be integrated directly by recognizing \( d(x^3+3x) = 3(x^2+1) \).

2. \( \int \frac{\sqrt{x^2+4}}{x} \, dx \) can be transformed into

\[
\int \frac{\sqrt{u^2-4}}{u} \, du \quad \text{where} \quad u = x + 4
\]

\[
= \int \frac{2(u^2-4)}{u^2-4} \, du + \int \frac{8}{u^2-4} \, du = 2u + 8 \left( \frac{1}{2} \right) \int \frac{1}{u-2} \, du
\]

\[
= 2\sqrt{x+4} + 4 \ln \left| \frac{x+4}{\sqrt{x+4}+2} \right| + C
\]