Approximate Integration

Since not every function can be integrated, so effective numerical integration techniques have been developed to approximate the area under a given \( f(x) \) \( (a \leq x \leq b) \). Besides, it may happen that we need to know the area under a hypothetical function that we only know its values at certain points collected over a survey or an experiment.

Let us recall that

\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{k=1}^n \frac{b-a}{n} f(x_k^*) \Delta x,
\]

where \( x_k^* \) is an arbitrarily chosen point in the interval \( [x_{k-1}, x_k] \). So one can only consider choosing the left or right-hand endpoints for the approximations:

\[
\int_a^b f(x) \, dx \approx L_n = \sum_{k=1}^n f(x_{k-1}) \Delta x \quad \text{i.e., } x_k^* = x_{k-1}
\]

Left endpoint approximation

\[
\int_a^b f(x) \, dx \approx R_n = \sum_{k=1}^n f(x_k) \Delta x \quad \text{i.e., } x_k^* = x_k
\]

Right endpoint approximation
But we can also choose the midpoint instead of the endpoints of each interval: \( x_k^* = \frac{x_{k-1} + x_k}{2} \) so that

\[
\int_a^b f(x) \, dx = M_n = \frac{b-a}{n} \sum_{k=1}^{n} f(x_k^*) \Delta x =
\]

\[
= \left[ f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*) \right] \Delta x
\]

**Midpoint Rule**

\[
\int_a^b f(x) \, dx = M_n = \Delta x \left[ f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n) \right]
\]

where \( \Delta x = \frac{b-a}{n} \)

and \( \bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) \) is midpoint of \([x_{i-1}, x_i]\)

**Remark** The Midpoint method appears to give a better approximation.
Trapezoidal Rule

\[
\int_a^b f(x) \, dx \approx \frac{\Delta x}{2} \left[ f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n) \right]
\]

where \( \Delta x = (b - a)/n \) and \( x_i = a + i \Delta x \).

The name of the Trapezoidal Rule comes from the observation that

\[
\frac{f(x_{k-1}) + f(x_k)}{2} \Delta x = \text{Area of a trapezium for } k = 1, \ldots, n
\]
More generally, we have

Thus the Trapezoidal Rule is to use trapeziums to approximate \( \int_a^b f(x) \, dx \). The left- or right-endpoint methods use rectangles to approximate \( \int_a^b f(x) \, dx \). So unless \( f(x) \) is a straight line, the Trapezoidal Rule should produce a better approximation.

Example: Apply both the Midpoint method and the trapezoidal rule to approximate \( \int_1^2 \frac{dx}{x} \).

Midpoint method: \([a, b] = [1, 2] \), \( n = 5 \), \( \Delta x = \frac{2-1}{5} = 0.2 \). So

\[
x_k = 1 + k \cdot \frac{0.2}{5}, \quad k = 0, 1, \ldots, 5,
\]

midpoints \( x_k + x_{k+1} = 1 + \frac{2k-1}{10}, k = 1, \ldots, 5 \).
\[
\text{Heme: } \int_{\frac{1}{2}}^{2} \frac{dx}{x} = \Delta x \sum_{k=1}^{5} f\left(\frac{k+2}{10}\right) = 0.2 \sum_{k=1}^{5} f\left(\frac{9+2k}{10}\right)
\]

\[
= 0.2 \left(\frac{1}{1+1.3+1.5+1.7+1.9}\right) \approx 0.691908
\]

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**Plot**

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**Method comparisons**

<table>
<thead>
<tr>
<th>method</th>
<th>result</th>
<th>absolute error</th>
<th>relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>left endpoint</td>
<td>0.745635</td>
<td>0.0524877</td>
<td>0.0757238</td>
</tr>
<tr>
<td>right endpoint</td>
<td>0.645635</td>
<td>0.0475123</td>
<td>0.0685457</td>
</tr>
<tr>
<td>midpoint</td>
<td>0.691908</td>
<td>0.00123929</td>
<td>0.00178792</td>
</tr>
<tr>
<td>trapezoidal rule</td>
<td>0.695635</td>
<td>0.00248774</td>
<td>0.00358905</td>
</tr>
<tr>
<td>Simpson's rule</td>
<td>0.69315</td>
<td>3.05013 × 10^{-6}</td>
<td>4.40041 × 10^{-6}</td>
</tr>
</tbody>
</table>
Trapezoidal Rule:

\[
\int_{1}^{2} \frac{dx}{x} = \frac{\Delta x}{2} \left( f(x_0) + 2 \sum_{k=1}^{n-1} f(x_k) + f(x_n) \right)
\]

\[
= \frac{1}{2} \sum_{k=1}^{4} \left( f(1) + \frac{k}{5} \right) + f(2)
\]

\[
= \frac{1}{10} \left( \frac{1}{2} + 2 \sum_{k=1}^{4} \frac{k+1}{5} \right)
\]

\[
= \frac{1}{10} \left( \frac{1}{2} + 2 \left( \frac{5}{6} + \frac{7}{5} + \frac{5}{8} + \frac{5}{9} \right) \right)
\]

\[
= 0.695659
\]
Exercise: We repeat the above example with $N = 10$.

\[
\Delta x = \frac{2 - 1}{10} = \frac{1}{10}
\]

\[
x_k = 1 + \frac{k}{10}, \quad k = 0, 1, \cdots, 10
\]

\[
\overline{x}_k = \frac{x_{k-1} + x_k}{2} = 1 + \frac{2k-1}{20}
\]

\[
k = 1, 2, \cdots, 10.
\]

Midpoint rule:

\[
\int_{1}^{2} f(x) \, dx = \Delta x \sum_{k=1}^{10} f\left(\overline{x}_k\right) = \frac{1}{10} \sum_{k=1}^{10} f\left(\frac{2k+1}{20}\right) = \frac{1}{10} \sum_{k=1}^{10} \frac{2}{2k+19}
\]

\[
\approx 0.692835
\]

Method comparisons

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</tr>
</thead>
<tbody>
<tr>
<td>left endpoint</td>
<td>0.718771</td>
<td>0.0256242</td>
<td>0.0369679</td>
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<tr>
<td>right endpoint</td>
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<td>midpoint</td>
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<td>0.000624223</td>
<td>0.000900563</td>
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<tr>
<td>Simpson's rule</td>
<td>0.693147</td>
<td>$1.94105 \times 10^{-7}$</td>
<td>$2.80035 \times 10^{-7}$</td>
</tr>
</tbody>
</table>
Trapezoidal Rule:

Plot

\[ \int_{1}^{2} \frac{dx}{x} = \frac{\sqrt{10}}{2} \left[ f(a) + 2 \sum_{k=1}^{n-1} f(x_k) + f(b) \right] \]

\[ = \frac{1}{20} \left[ f(1) + 2 \sum_{k=1}^{9} f(4 + \frac{k}{10}) + f(2) \right] \]

\[ = \frac{1}{20} \left[ 1 + 2 \times \frac{9}{10} \left( \frac{1}{0 + \frac{1}{2}} + \frac{1}{2} \right) \right] = 0.693771 \]

Error Analysis

3. Error Bounds: Suppose $|f''(x)| \leq K$ for $a \leq x \leq b$. If $E_T$ and $E_M$ are the errors in the Trapezoidal and Midpoint Rules, then

\[ |E_T| \leq \frac{K(b - a)^3}{12n^2} \quad \text{and} \quad |E_M| \leq \frac{K(b - a)^3}{24n^2} \]

Remarks:
1. The $f''(x)$ measures the concavity (i.e., how flat the curve)
2. The $E_M$ may be smaller than $E_T$ by a factor of two
3. When we double the partition points from \( n \) to \( 2n \), the error drops by a factor of \( \frac{(2n)^2}{n^2} = 4 \) in both the Midpoint or Trapezoidal rules.

The remark 2 can also be seen from the following figure:

Example: Estimate the error bounds in the first example.

\[
\frac{f(x)}{x^6}, \quad \left| f''(x) \right| = \frac{2}{x^5} = \frac{2}{x^5} \leq 2 \quad (1 \leq x \leq 2)
\]

\[
|E_M| \leq \frac{2(2-1)^3}{24(5)^2} = \frac{2}{24 \times 25} = \frac{1}{300} \approx 0.000333
\]

Remark: Compare the actual error in the Mathematica table above.

Exercise: Compute the error for the \( E_M \).
Example: How large should we take $n$ in the trapezoidal rule in order to guarantee that the approximation to $\int_{1}^{2} \frac{dx}{x}$ are accurate to within 0.0001?

We want to solve for the inequality for $n$

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} = \frac{2(2-1)^3}{12n^2} = \frac{1}{6n^2} \leq 0.0001.$$ 

That is,

$$n^2 \geq \frac{1}{6 \times 0.0001} = \frac{1}{0.0006}.$$ 

OR

$$n \geq \frac{1}{\sqrt{0.0006}} \approx 40.8.$$ 

Can take $n = 41$. 
