6.3 Volumes by Cylindrical Shells

We consider volumes that are difficult or even impossible to find with the previous section.

Example Find the volume of the solid obtained by rotating about the
the region \( y = 2x^2 - x^3 \) and \( y = 0 \) about the \( y \)-axis.

If we apply the method used in the previous section to find area of vertical thin slabs, then given \( x \) both inner and outer radii are written in terms of \( y \) coordinate. But this would be difficult as we need to relate the \( x \) in terms of \( y \).

But solving the cubic equation \( y = 2x^2 - x^3 \) is not recommended.

We sum the volume \( V \) by cylindrical shells

\[
\Delta V = \pi (r_f^2 - r_i^2) \, h
\]

\[
= \pi (r_2^2 - r_1^2) \, h
\]

\[
= \pi \cdot 2 \Delta r \cdot (r_2 - r_1) \, h
\]

\[
= 2 \pi \Delta r \cdot \text{circumference} \cdot \text{height}
\]

since we denote \( \Delta r = r_2 - r_1 \).
One can consider the following set up where even the height $h=f(x)$ is a function of $x$.

The volume is then given by

$$V = \lim_{n \to \infty} \sum_{k=1}^{n} \Delta V_k = \lim_{n \to \infty} \sum_{k=1}^{n} 2\pi x_k^* f(x_k^*) \Delta x$$

$$= \int_{a}^{b} 2\pi x f(x) \, dx$$

This would give

Let us return to our original problem.
We identify \( a = 0, \ b = 2, \ y = f(x) = 2x^2 - x^3 \) so that

\[
V = \lim_{n \to \infty} \sum_{k=1}^{n} 2\pi x_k f(x_k) \Delta x = \int_{0}^{2} 2\pi x f(x) \, dx
\]

\[
= \int_{0}^{2} 2\pi x (2x^2 - x^3) \, dx = 2\pi \int_{0}^{2} 2x^3 - x^4 \, dx = \frac{16}{3} \pi.
\]

**Example** Find the volume of the solid obtained by rotating about the \( y \)-axis the region between \( y = x \) and \( y = x^2 \).

\[
\Delta V = (\text{circumference of } y) \times \text{height}
\]

\[
= 2\pi x (x - x^2)
\]

\[
\Rightarrow \ V = \int_{0}^{1} 2\pi (x^2 - x^3) \, dx = \frac{7}{6}.
\]
Exercise 6.3 Q2

Let $S$ be the solid obtained by rotating the region on the left about the y-axis. Find the volume of the $S$ by cylindrical shells. Is this method preferable to slicing? Explain.

\[ \Delta V = (\text{circumference}) \times \text{height} \times dx = 2\pi x \sin(x^2) \, dx \]

\[ V = \int_0^{\sqrt{\pi}} 2\pi x \sin(x^2) \, dx = -\pi \cos x^2 \bigg|_0^{\sqrt{\pi}} = -\pi (\cos \pi - \cos 0) = 2\pi. \]

Horizontal slicing method would be very difficult. Why?