Part I: MC Answers

White Version

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Part II: Long Questions

1. [14 pts] The shape of a container is the same as the surface of revolution obtained by rotating the curve \( x = y^2 \) about the \( y \)-axis, where \( 0 \leq x \leq 100 \) (in meters). Suppose water flows into the container and stops flowing in just when 50% of the volume is filled. (Water density =1000 kg/m\(^3\), and \( g = 9.8 \) m/s\(^2\).)

   (a) Find the minimum work required to pump the water back to the top of the container. [9 pts]

   **Answer:** Let \( k \) be the depth of water in the container when it is 50% filled. Then

   \[
   \int_0^k \pi y^4 \, dy = \frac{1}{2} \int_0^{10} \pi y^4 \, dy
   \]

   \[
   \pi \cdot \frac{k^5}{5} = \frac{1}{2} \pi \frac{10^5}{5}
   \]

   \[
   k^5 = \frac{10^5}{2} \iff k = \frac{10}{\sqrt[5]{2}}
   \]

   The work required to pump the water to the top of the container is

   \[
   \int_0^{\frac{10}{\sqrt[5]{2}}} \rho g \pi y^4 (10 - y) \, dy
   \]

   \[
   = 9800\pi \int_0^{\frac{10}{\sqrt[5]{2}}} (10y^4 - y^5) \, dy
   \]

   \[
   = 9800\pi \left[ 2y^5 - \frac{y^6}{6} \right]_0^{\frac{10}{\sqrt[5]{2}}}
   \]

   \[
   = 9.8 \cdot 10^5 \pi \left( 1 - \frac{10}{12\sqrt[5]{2}} \right)
   \]

   (b) Express by a definite integral the hydrostatic force on the inside surface of the container when 50% of its volume is filled. [5 pts]

   **Answer:** The hydrostatic force on the surface of the container is

   \[
   \int_0^k \rho g (k - y) 2\pi y^2 \sqrt{1 + \left( \frac{dy^2}{dy} \right)^2} \, dy
   \]

   \[
   = \int_0^{\frac{10}{\sqrt[5]{2}}} (9.8 \cdot 1000)(2\pi)\left( \frac{10}{\sqrt[5]{2}} - y \right)y^2 \sqrt{1 + 4y^2} \, dy
   \]
2. [14 pts] Curve I on the $xy$-plane is defined by the polar equation $r = 1 - 3 \cos \theta$ whose graph is given below.

(a) Consider polar Curve II defined by the polar equation $r = 3 + \cos(2 \theta)$. Fill in the exact radial coordinates of some points on this curve in the following table, and then sketch Curve II together with Curve I in the given figure.

<table>
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<th>0</th>
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<tbody>
<tr>
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<td></td>
</tr>
<tr>
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<td>3</td>
<td>2</td>
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(Plot Curve II in this plane.)

(b) Find the area of the region which lies inside Curve II, but does not overlap with any part of the region enclosed by Curve I. [8 pts]

Answer: The curves intersect when $1 - 3 \cos \theta = 3 + \cos 2 \theta$; i.e.,

$$\cos 2 \theta + 3 \cos \theta + 2 = 0$$

$$2 \cos^2 \theta + 3 \cos \theta + 1 = (2 \cos \theta + 1)(\cos \theta + 1) = 0$$

$$\theta = \pm \frac{2\pi}{3}, \pi.$$ Note that the curves are symmetric with respect to the $x$-axis. Curve I hits the origin when $\cos \theta = \frac{1}{3}$; e.g., when $\theta = \cos^{-1} \frac{1}{3}$.

The area of the region wanted is:

$$2 \int_0^{2\pi} \frac{1}{2}(3 + \cos 2 \theta)^2 d\theta - 2 \int_{\cos^{-1} \frac{1}{3}}^{2\pi} \frac{1}{2}(1 - 3 \cos \theta)^2 d\theta$$

$$= \int_0^{2\pi} (9 + 6 \cos 2 \theta + \cos^2 2 \theta) d\theta - \int_{\cos^{-1} \frac{1}{3}}^{2\pi} (1 - 6 \cos \theta + 9 \cos^2 \theta) d\theta$$

$$= \left[ \frac{19}{2} \theta + 3 \sin 2 \theta \right]_0^{2\pi} + \frac{1}{2} \int_0^{2\pi} \cos 4 \theta d\theta - \left[ \frac{11}{2} \theta - 6 \sin \theta \right]_{\cos^{-1} \frac{1}{3}}^{2\pi} - \frac{9}{2} \int_{\cos^{-1} \frac{1}{3}}^{2\pi} \cos 2 \theta d\theta$$

$$= \frac{8\pi}{3} + 3 \sin \frac{4\pi}{3} + 6 \sin \frac{2\pi}{3} + \frac{11}{2} \cos^{-1} \frac{1}{3} - 6 \sin \cos^{-1} \frac{1}{3} + \frac{9}{4} \sin 2 \theta \bigg|_{\cos^{-1} \frac{1}{3}}^{2\pi}$$

$$= \frac{8\pi}{3} + \frac{43}{16} \sqrt{3} + \frac{11}{2} \cos^{-1} \frac{1}{3} - 6 \sin \cos^{-1} \frac{1}{3} + \frac{9}{4} \sin 2 \cos^{-1} \frac{1}{3}$$

$$= \frac{8\pi}{3} + \frac{43}{16} \sqrt{3} + \frac{11}{2} \cos^{-1} \frac{1}{3} - 3 \sqrt{2}$$

Note that $\sin \cos^{-1} \frac{1}{3} = \frac{2\sqrt{2}}{3}$. 