HYPERGEOMETRIC EQUATION AND MONODROMY GROUPS

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Abstract. We shall introduce the basics of classical Gauss hypergeometric equation and to study its monodromy problem. Historically, Riemann was the first one who worked out the monodromy group of the DE in 1857.

1. Fuchsian-type Differential Equations

Theorem 1.1. *Let $a_1, a_2, \cdots a_n$ and ∞ be the regular singular points of*

$$\frac{d^2w}{dz^2} + p(z)\frac{dw}{dz} + q(z)w = 0. \tag{1.1}$$

Then we have

$$p(z) = \sum_{k=1}^{n} \frac{A_k}{z - a_k}, \quad q(z) = \sum_{k=1}^{n} \Big(\frac{B_k}{(z-a_k)^2} + \frac{C_k}{z - a_k} \Big), \tag{1.2}$$

and the indicial equation at each a_k is given by

$$\alpha^2 + (A_k - 1)\alpha + B_k = 0.$$

Suppose $\{\alpha_k, \alpha'_k\}$ and $\{\alpha_\infty, \alpha'_\infty\}$ denote, respectively, the exponents of the corresponding indicial equations at $a_1, a_2, \cdots a_n$ and ∞. Then we have

$$(\alpha_\infty + \alpha'_\infty) + \sum_{k=1}^{n} (\alpha_k + \alpha'_k) = n - 1,$$

Proof. Since $p(z)$ has at most a first order pole at each of the points $a_1, a_2, \cdots a_n$, so the function $\phi(z)$ defind in

$$p(z) = \sum_{k=1}^{n} \frac{A_k}{z - a_k} + \varphi(z)$$

where $\varphi(z)$ is an entire function in $\mathbb{C}$, and where the A_k is the residue of $p(z)$ at a_k $(k = 1, 2, \cdots n)$. On the other hand, the ∞ is also a regular singular point of $p(z)$, it follows from Proposition 3.2 in part I and which states ∞ is regular singular iff zp and z^2q are analytic at ∞. That is, $p(z) = O(1/z)$. So $\varphi(z) \to 0$ and hence must be the constant zero by applying Liouville's theorem from complex analysis. Hence

$$p(z) = \sum_{k=1}^{n} \frac{A_k}{z - a_k}.$$

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Similarly, we have

$$q(z)=\sum_{k=1}^{n}\Big(\frac{B_k}{(z-a_k)^2}+\frac{C_k}{z-a_k}\Big)+\psi(z),$$

where $\psi(z)$ is entire function. But $z^2q(z)$ is analytic at ∞, so $q(z)=O(1/z^2)$ as $z\to\infty$. This shows that $\psi(z)\equiv 0$. The Proposition 3.2 (in Part I) again implies that

$$\sum_{k=1}^{n}C_k=0.$$

Considering the Taylor expansions around the regular singular point a_k

$$(z-a_k)p(z)=\sum_{j=0}^{\infty}p_j\cdot(z-a_k)^j,\quad (z-a_k)^2q(z)=\sum_{j=0}^{\infty}q_j\cdot(z-a_k)^j$$

and

$$w(z)=\sum_{j=0}^{\infty}a_j(z-a_k)^{\alpha+j}$$

yields

$$\alpha^2+(A_k-1)\alpha+B_k=0.$$

for $k=1,\,2,\,\cdots,n$. We now deal with the indicial equation at $z=\infty$. Since the ∞ is also a regular singular point, so the Proposition 3.2 (in Part I) implies that

$$\begin{aligned}\lim_{t\to 0}t\Big(\frac{2}{t}-\frac{1}{t^2}p\Big(\frac{1}{t}\Big)\Big)&=2-\lim_{z\to\infty}zp(z)\\&=2-\lim_{z\to\infty}z\left\{\sum_{k=1}^{n}\frac{\prod_{\substack{j=1\\ j\neq k}}^{n}A_k(z-a_j)}{\prod_{j=1}^{n}(z-a_j)}\right\}\\&=2-\lim_{z\to\infty}\frac{\big(\sum_{k=1}^{n}A_k\big)z^n+O(z^{n-1})}{z^n+O(z^{n-1})}\\&=2-\sum_{k=1}^{n}A_k.\end{aligned}\tag{1.3}$$

Similarly, we have

$$\begin{aligned}\lim_{t\to 0}t^2\cdot\frac{1}{t^4}q\Big(\frac{1}{t}\Big)&=\lim_{z\to\infty}z^2q(z)\\&=\lim_{z\to\infty}z^2\sum_{k=1}^{n}\Big(\frac{B_k}{(z-a_k)^2}+\frac{C_k}{z-a_k}\Big)\\&=\sum_{k=1}^{n}B_k+\lim_{z\to\infty}z\sum_{k=1}^{n}\frac{C_k}{1-a_k/z}\\&=\sum_{k=1}^{n}B_k+\lim_{z\to\infty}z\sum_{k=1}^{n}C_k\big[1+\frac{a_k}{z}+O\big(\frac{1}{z^2}\big)\big]\\&=\sum_{k=1}^{n}B_k+0+\sum_{k=1}^{n}a_kC_k,\end{aligned}\tag{1.4}$$

because $\sum_{k=1}^{n} C_k = 0$. The indicial equation for the point $z = 0$ is of the form

$$\alpha^2 + (p_0 - 1)\alpha + q_0 = 0,$$

where $p_0 = \lim_{z\to 0} zp(z)$ and $q_0 = \lim_{z\to 0} z^2 q(z)$. Thus, the point at infinity is given by

$$\alpha^2 + \Big(1 - \sum_{k=1}^{n} A_k\Big)\alpha + \sum_{k=1}^{n}(B_k + a_k C_k) = 0.$$

Let α_k, α'_k be the exponents of the indicial equation $\alpha^2 + (A_k - 1)\alpha + B_k = 0$ for $k = 1, 2, \cdots, n$, and α_∞, α'_∞ be the exponents of the indicial equation at ∞. Thus we deduce

$$\sum_{k=1}^{n}(\alpha_k + \alpha'_k) = -\sum_{k=1}^{n}(A_k - 1) = \sum_{k=1}^{n}(1 - A_k) = n - \sum_{k=1}^{n} A_k,$$

and $\alpha_\infty + \alpha'_\infty = 1 - \sum_{k=1}^{n} A_k$. Hence we deduce

$$\sum_{k=1}^{n}(\alpha_k + \alpha'_k) + (\alpha_\infty + \alpha'_\infty) = n - 1,$$

as required. □

Exercise 1.2 (Klein)**.** We note that if ∞ is an ordinary point, then show that

$$p(z) = \frac{2}{z} + \frac{p_2}{z^2} + \cdots,$$

and

$$q(z) = \frac{q_4}{z^4} + \frac{q_5}{z^5} + \cdots.$$

Suppose the ∞ in Theorem 5.1 is an ordinary point. Then show that the following identities hold.

$$\sum_{k=1}^{n} A_k = 2,$$
$$\sum_{k=1}^{n} C_k = 0,$$
$$\sum_{k=1}^{n}(B_k + a_k C_k) = 0,$$
$$\sum_{k=1}^{n}(2a_k B_k + a_k^2 C_k) = 0.$$

2. Differential Equations with Three Regular Singular Points

We again consider differential equation in the form

$$\frac{d^2 w}{dz^2} + p(z)\frac{dw}{dz} + q(z)w = 0. \tag{2.1}$$

The following statement was first written down by Papperitz in 1885. See [9]

Theorem 2.1. *Let the differential equation (2.1) has regular singular points at ξ, η and ζ in $\mathbb{C}$. Suppose the α, α'; β, β'; γ, γ' are, respectively, the exponents at the regular singular points ξ, η and ζ. Then the differential equation must assume the form*

$$\begin{aligned}\frac{d^2w}{dz^2}+\Big\{\frac{1-\alpha-\alpha'}{z-\xi}+\frac{1-\beta-\beta'}{z-\eta}+\frac{1-\gamma-\gamma'}{z-\zeta}\Big\}\frac{dw}{dz}-\\ -\Big\{\frac{\alpha\alpha'}{(z-\xi)(\eta-\zeta)}+\frac{\beta\beta'}{(z-\eta)(\zeta-\xi)}+\frac{\gamma\gamma'}{(z-\zeta)(\xi-\eta)}\Big\}\times\\ \times\frac{(\xi-\eta)(\eta-\zeta)(\zeta-\xi)}{(z-\xi)(z-\eta)(z-\zeta)}\,w=0.\end{aligned}\tag{2.2}$$

Proof. The equation has regular singular points at ξ, η and ζ in $\mathbb{C}$. Thus the functions

$$P(z)=(z-\xi)(z-\eta)(z-\zeta)\,p(z),$$

and

$$Q(z)=(z-\xi)^2(z-\eta)^2(z-\zeta)^2\,q(z)$$

are both entire functions (complex differentiable everywhere). Besides, since the point at "∞" is an analytic point for both $p(z)$ and $q(z)$ (since we have assumed that none of the three singular points $\{\xi,\eta,\zeta\}$ is at ∞), so we deduce from the Exercise 1.2 that the functions that $P(z)$ and $Q(z)$ have degree two. That is,

$$p(z)=\frac{P(z)}{(z-\xi)(z-\eta)(z-\zeta)}=\frac{A}{z-\xi}+\frac{B}{z-\eta}+\frac{C}{z-\zeta},$$

where $A+B+C=2$ (by Exercise 1.2 again). Similarly,

$$\begin{aligned}(z-\xi)(z-\eta)(z-\zeta)q(z)&=\frac{Q(z)}{(z-\xi)(z-\eta)(z-\zeta)}\\ &=\frac{D}{z-\xi}+\frac{E}{z-\eta}+\frac{F}{z-\zeta}.\end{aligned}$$

Substitute $w(z)=\sum_{k=0}^{\infty}w_k(z-\xi)^{\alpha+k}$ into the differential equation and note the coefficient $q(z)$ assumes the form

$$(z-\xi)^2q(z)=\frac{D}{(z-\eta)(z-\zeta)}+\frac{E(z-\xi)}{(z-\eta)^2(x-\zeta)}+\frac{F(z-\xi)}{(z-\eta)(z-\zeta)^2}.$$

So $q_0=\frac{D}{(\xi-\eta)(\xi-\zeta)}$. Thus the indicial equation at $z=\xi$ is given by

$$\alpha(\alpha-1)+A\alpha+\frac{D}{(\xi-\eta)(\xi-\zeta)}=\alpha^2+(A-1)\alpha+\frac{D}{(\xi-\eta)(\xi-\zeta)}=0,$$

where $\alpha+\alpha'=-(A-1)=1-A$ or $A=1-\alpha-\alpha'$, and

$$\alpha\alpha'=\frac{D}{(\xi-\eta)(\xi-\zeta)},$$

or $D=(\xi-\eta)(\xi-\zeta)\alpha\alpha'$. Similarly, we have , at the regular singular points η, ζ that $B=1-\beta-\beta'$ and $C=1-\gamma-\gamma'$, and hence

$$E=(\eta-\xi)(\eta-\zeta)\beta\beta',\qquad F=(\zeta-\xi)(\zeta-\eta)\gamma\gamma'.$$

This yields the differential equation of the required form. □

Remark 2.2. We note that the constraint $A+B+C=2$ implies that

$$\alpha+\alpha'+\beta+\beta'+\gamma+\gamma'=1.$$

Example 2.1. Suppose that a second order linear differential equation has only one regular singular point at $z=0$ and infinity is an ordinary point. Show that the equation is of the form $zw''+2w'=0$.

Proof. Let the equation be in the form $w''+p(z)w'+q(z)w=0$. Since $z=0$ is a regular singular point. So $P(z)=zp(z)$ is an entire function. But $z=\infty$ is an ordinary point, so $P(z)=zp(z)=z\big(\frac{A}{z}+\frac{p_2}{z^2}+\cdots\big)=A+\frac{p_2}{z}+\cdots$ as $z\to\infty$ by Exercise 1.2. We conclude that P is the constant A by Liouville's theorem. We also deduce that $A=2$ since $p(z)=\frac{2}{z}+\frac{p_2}{z^2}+\cdots$. Similarly, we have $Q(z)=z^2q(z)$ to be an entire function. But $z=\infty$ is an ordinary point, so $Q(z)=z^2q(z)=z^2\big(\frac{q_4}{z^4}+\frac{q_5}{z^5}+\cdots\big)=\frac{q_4}{z^2}+\frac{q_4}{z^3}+\cdots\to 0$ as $z\to\infty$ by Exercise 1.2 again. Thus $Q(z)\equiv 0$ by Liouville's theorem again. Hence $q(z)\equiv 0$. We conclude that $w''+\frac{2}{z}w'+0=0$. □

Exercise 2.3. Suppose a second order linear differential equation has both the origin and the infinity to be regular singular points, and that their respective exponents are $\{\alpha,\,\alpha'\}$ and $\{\alpha_\infty,\,\alpha'_\infty\}$. Derive the exact form of the differential equation and show that $\alpha+\alpha'+\alpha_\infty+\alpha'_\infty=0$ and $\alpha\cdot\alpha'=\alpha_\infty\cdot\alpha'_\infty$ hold.

If we now choose $\eta=\infty$ in the differential equation in Theorem 2.1, then we easily obtain

$$\begin{aligned}&\frac{d^2w}{dz^2}+\Big\{\frac{1-\alpha-\alpha'}{z-\xi}+\frac{1-\gamma-\gamma'}{z-\zeta}\Big\}\frac{dw}{dz}+\\&\qquad+\Big\{\frac{\alpha\alpha'(\xi-\zeta)}{z-\xi}+\beta\beta'+\frac{\gamma\gamma'(\zeta-\xi)}{z-\zeta}\Big\}\times\frac{w}{(z-\xi)(z-\zeta)}=0.\end{aligned}\tag{2.3}$$

For convenience we could set $\xi=0$ and $\zeta=1$. So this differential equation will be further reduced to the form

$$\begin{aligned}&\frac{d^2w}{dz^2}+\Big\{\frac{1-\alpha-\alpha'}{z}+\frac{1-\gamma-\gamma'}{z-1}\Big\}\frac{dw}{dz}+\\&\qquad+\Big\{\frac{\alpha\alpha'(-1)}{z}+\beta\beta'+\frac{\gamma\gamma'(1-0)}{z-1}\Big\}\times\frac{w}{z(z-1)}=0.\end{aligned}\tag{2.4}$$

We note that the exponent pair at each of the singularities is exactly the same as before.

Exercise 2.4. *Show that the above equation can be written in the from*

$$\begin{aligned}&\frac{d^2w}{dz^2}+\Big\{\frac{1-\alpha-\alpha'}{z}+\frac{1-\gamma-\gamma'}{z-1}\Big\}\frac{dw}{dz}+\\&\qquad+\Big\{\frac{\alpha\alpha'}{z^2}+\frac{\gamma\gamma'}{(z-1)^2}+\frac{\beta\beta'-\alpha\alpha'-\gamma\gamma'}{z(z-1)}\Big\}w=0.\end{aligned}\tag{2.5}$$

Before we embark on more structural results, we need the following result.

Proposition 2.1. Let

$$\frac{d^2w}{dz^2} + p(z)\frac{dw}{dz} + q(z)w = 0 \tag{2.6}$$

to have a regular singular point at $z_0 = 0$ with exponent $\{\alpha,\, \alpha'\}$. Then the function $f(z)$ defined by $w(z) = z^\lambda f(z)$ satisfies the differential equation

$$\frac{d^2f}{dz^2} + \Big(p(z) + \frac{2\lambda}{z}\Big)\frac{df}{dz} + \Big(q(z) + \frac{\lambda p(z)}{z} + \frac{\lambda(\lambda-1)}{z^2}\Big)f = 0, \tag{2.7}$$

with regular singular point also at $z = 0$ but the exponent pair is $\{\alpha - \lambda,\, \alpha' - \lambda\}$. If $z = \infty$, then the equation (2.7) has exponent pair $\{\alpha + \lambda,\, \alpha' + \lambda\}$ instead.

Proof. Let $w(z) = z^\lambda f(z)$. We leave the verification of the equation (2.7) as an exercise. Substitute the series

$$zp(z) = \sum_{k=0}^{\infty} p_k z^k, \qquad z^2q(z) = \sum_{k=0}^{\infty} q_k z^k, \qquad f(z) = \sum_{k=0}^{\infty} a_k z^{\alpha+k}$$

into the differential equation (2.7) and consider the indicial equation:

$$\alpha(\alpha-1) + \big(p_0 + 2\lambda\big)\alpha + q_0 + \lambda p_0 + \lambda(\lambda-1) = 0,$$

or

$$\alpha^2 + \big(p_0 - 1 + 2\lambda\big)\alpha + q_0 + \lambda p_0 + \lambda(\lambda-1) = 0.$$

Let $\{\tilde\alpha,\, \tilde\alpha'\}$ be the exponent pair for the equation (2.7) at $z = 0$. But then $\tilde\alpha + \tilde\alpha' = 1 - p_0 - 2\lambda = \alpha + \alpha' - 2\lambda$, and

$$\begin{aligned}\tilde\alpha\tilde\alpha' &= q_0 + \lambda p_0 + \lambda(\lambda-1)\\ &= \alpha\alpha' - \lambda(\alpha+\alpha') + \lambda^2\\ &= (\alpha-\lambda)(\alpha'-\lambda).\end{aligned}$$

Solving the above two algebraic equations proves the first part of the proposition. □

Exercise 2.5. Complete the proof of the above Proposition of the transformation to ∞ and the change of the exponent pair.

Remark 2.6. If we apply the transformation $w(z) = (z-1)^\lambda f(z)$ to the differential equation (2.6), then the resulting equation (2.7) would still have $\{0,\, 1,\, \infty\}$ as regular singular points, but the exponent pair at $z = 1$ is now $\{\beta - \lambda,\, \beta' - \lambda\}$ instead. The idea could be apply to a more general situation. Suppose the differential equation (2.6) has $z = \xi$ as a regular singular point with exponent pair $\{\alpha,\, \alpha'\}$, then the resulting differential equation after the transformation $w(z) = (z-\xi)^\lambda f(z)$ would still has $z = \xi$ as a regular singular point but now the corresponding exponent pair is $\{\alpha - \lambda,\, \alpha' - \lambda\}$.

3. Mondromy

We consider linear differential equation with finite number of *singularities* in the one-dimensional projective space $\mathbb{CP}^1$, that is $\mathbb{C} \cup \{\infty\}$.

Definition 3.1. Let D be simply connected. Let $b \in D$.

(1) A *loop* γ in D
$$\gamma : I = [0,\, 1] \to D$$
such that $\gamma(0) = b = \gamma(1)$. The b is called the *base point* of γ. Two loops γ_0, γ_1 in D are *homotopy (equivalent)*, denoted by $\gamma_0, \simeq \gamma_1$, if γ_0 can be deformed continusely to γ_1 and vice-versa (but keeping the base point fixed).

(2) Let $L(D,\, b)$ be the set of loops with a base point at b. Then $\simeq$ defines an equivalence relation on $L(D,\, b)$.

(3) The set of all equivalence classes $[\gamma]$ of loops defined above forms a group $\pi_1(D,\, b) = L(D,\, b)/\simeq$, called the *fundamental group.*We define the product of two loops $\gamma_0 \cdot \gamma_1$ to be a loop such that γ_1 *follows* γ_0. For each loop γ one can define its inverse γ^{-1} by $\gamma^{-1}(t) = \gamma(1-t)$ and an *identity* (constant) loop $c : I \to D$ such that $c(t) \equiv b$ for all t. It is easy to verify that one carry this struture to their equivalence classes. Hence one can really make $\pi_1(D,\, b) = L(D,\, b)/\simeq$ to be a group: with $\alpha = [\gamma] \in \pi_1(D,\, b)$, $\alpha^{-1} = [\gamma^{-1}]$, etc.

4. Riemann's $P-$Scheme

The following fomulation of solution of Gauss hypergeometric equation appeared to be first put forward by B. Riemann (1826–1866) in 1857 [7]. We shall follow closely Riemann's exposition.

Definition 4.1. Following Riemann, we write the solution w of the equation (2.2) in Riemann's $P-$symbol:

$$w(z) = P\left\{\begin{array}{cccc} \xi & \eta & \zeta & \\ \alpha & \beta & \gamma\,; & z \\ \alpha' & \beta' & \gamma' & \end{array}\right\}. \tag{4.1}$$

Remark 4.1. *We see immediately that the meaning of Riemann's $P-$function remains unchanged if we permuate the first three columns or interchange any of the two exponents for each of the three points $\{\xi,\, \eta,\, \zeta\}$.*

Thus a solution of equation (2.3) can be denoted by

$$w(z) = P\left\{\begin{array}{cccc} \xi & \infty & \zeta & \\ \alpha & \beta & \gamma\,; & z \\ \alpha' & \beta' & \gamma' & \end{array}\right\},$$

while that of (2.5) can be denoted by

$$w(z) = P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ \alpha & \beta & \gamma\,; & z \\ \alpha' & \beta' & \gamma' & \end{array}\right\}. \tag{4.2}$$

It follows from Proposition 2.1 that

Exercise 4.2.

$$z^p(z-1)^q P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ \alpha & \beta & \gamma\,; & z \\ \alpha' & \beta' & \gamma' & \end{array}\right\} = P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ \alpha+p & \beta-p-q & \gamma+q\,; & z \\ \alpha'+p & \beta'-p-q & \gamma'+q & \end{array}\right\}. \tag{4.3}$$

Similarly,

Exercise 4.3.

(4.4)

$$z^{-p}(z-1)^{-q} P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ \alpha & \beta & \gamma\,; & z \\ \alpha' & \beta' & \gamma' & \end{array}\right\} = P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ \alpha-p & \beta+p+q & \gamma-q\,; & z \\ \alpha'-p & \beta'+p+q & \gamma'-q & \end{array}\right\}.$$

Exercise 4.4. Prove by transforming the differential equation that

$$P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ \alpha & \beta & \gamma\,; & z \\ \alpha' & \beta' & \gamma' & \end{array}\right\} = P\left\{\begin{array}{cccc} 1 & 0 & \infty & \\ \alpha & \beta & \gamma\,; & \dfrac{1}{1-z} \\ \alpha' & \beta' & \gamma' & \end{array}\right\} \tag{4.5}$$

In general, we have

Exercise 4.5.

$$\left(\frac{z-\xi}{z-\eta}\right)^p\left(\frac{z-\zeta}{z-\eta}\right)^q P\left\{\begin{array}{cccc} \xi & \eta & \zeta & \\ \alpha & \beta & \gamma\,; & z \\ \alpha' & \beta' & \gamma' & \end{array}\right\} = P\left\{\begin{array}{cccc} \xi & \eta & \zeta & \\ \alpha+p & \beta-p-q & \gamma+q\,; & z \\ \alpha'+p & \beta'-p-q & \gamma'+q & \end{array}\right\}.$$

Exercise 4.6. Let $M(z) = \frac{az+b}{cz+d}$, where $ad - bc \neq 0$, be a Möbius transformation. Let $M(\xi) = \xi_1$, $M(\eta) = \eta_1$, $M(\zeta) = \zeta_1$. Show that

$$P\left\{\begin{array}{cccc} \xi & \eta & \zeta & \\ \alpha & \beta & \gamma\,; & z \\ \alpha' & \beta' & \gamma' & \end{array}\right\} = P\left\{\begin{array}{cccc} \xi_1 & \eta_1 & \zeta_1 & \\ \alpha & \beta & \gamma\,; & z_1 \\ \alpha' & \beta' & \gamma' & \end{array}\right\}$$

where $z_1 = M(z)$.

However, Riemann's original exposition is different from the above line of development. In fact, he considered funcitons that satisfy the following properties:

(1) Every branch of (2.2) is finite, single-valued and holomorphic (analytic) at every z except at three singularities a, b, c;
(2) Any three branches are linearly connected, or linearly dependent that is,
$$c'P' + c''P'' + c'''P''' = 0;$$
(3) At any of the three singularities, such as $z = a$ for example, there are two branches, denoted by $\{P^{(\alpha)}, P^{(\alpha')}\}$, corresponds to the two exponents $\{\alpha, \alpha'\}$, such that $(z-\xi)^\alpha P^{(\alpha)}$ and $(z-\xi)^{\alpha'} P^{(\alpha')}$ remain single-valued. Similarly, we have $\{P^{(\beta)}, P^{(\beta')}\}$ for $\{\beta, \beta'\}$ and finally $\{P^{(\gamma)}, P^{(\gamma')}\}$ for $\{\gamma, \gamma'\}$;
(4) that the six exponents $\{\alpha, \alpha', \beta, \beta', \gamma, \gamma'\}$ are such that none of the differences $(\alpha - \alpha')$, $(\beta - \beta')$, $(\gamma - \gamma')$ is an integer, and further that they all satisfy
$$\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1.$$

Thus the above exercises follow easily from these four requirements. Moreover, we automatically obtain that any two of the following six functions generated by (4.2) are equal to each other:

$$P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ \alpha & \beta & \gamma\,; & z \\ \alpha' & \beta' & \gamma' & \end{array}\right\},\quad P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ \beta & \gamma & \alpha\,; & \dfrac{1}{1-z} \\ \beta' & \gamma' & \alpha' & \end{array}\right\},\quad P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ \gamma & \alpha & \beta; & 1-\dfrac{1}{z} \\ \gamma' & \alpha' & \beta' & \end{array}\right\}$$

(4.6)

$$P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ \alpha & \gamma & \beta\,; & \dfrac{z}{z-1} \\ \alpha' & \gamma' & \beta' & \end{array}\right\},\quad P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ \beta & \alpha & \gamma\,; & \dfrac{1}{z} \\ \beta' & \alpha' & \gamma' & \end{array}\right\},\quad P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ \gamma & \beta & \alpha; & 1-z \\ \gamma' & \beta' & \alpha' & \end{array}\right\}.$$

The six Möbius transformations permute the three singularities in six possible ways.

4.1. **Some reduced forms.** The Exercise 4.4 implies that by choosing $p = \alpha$ and $q = \gamma$ that we would obtain

$$\begin{aligned} P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ \alpha & \beta & \gamma\,; & z \\ \alpha' & \beta' & \gamma' & \end{array}\right\} &= z^{\alpha}(1-z)^{\gamma}\,P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ \alpha-\alpha & \alpha+\beta+\gamma & \gamma-\gamma\,; & z \\ \alpha'-\alpha & \alpha+\beta'+\gamma & \gamma'-\gamma & \end{array}\right\} \\ &= z^{\alpha}(1-z)^{\gamma}\,P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ 0 & \alpha+\beta+\gamma & 0\,; & z \\ \alpha'-\alpha & \alpha+\beta'+\gamma & \gamma'-\gamma & \end{array}\right\}. \end{aligned} \tag{4.7}$$

So one can ensure that exactly one series solution at each of the regular singularities $z = 0$ and $z = 1$ is analytic there. By exchanging the exponent pair α and α' in the above derivation suggests that one can obtain two different representations. The new one being

$$\begin{aligned} P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ \alpha & \beta & \gamma & ;\ z \\ \alpha' & \beta' & \gamma' & \end{array}\right\} &= z^{\alpha'}(1-z)^{\gamma}\,P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ \alpha-\alpha' & \alpha'+\beta+\gamma & \gamma-\gamma & ;\ z \\ \alpha'-\alpha' & \alpha'+\beta'+\gamma & \gamma'-\gamma & \end{array}\right\} \\ &= z^{\alpha'}(1-z)^{\gamma}\,P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ \alpha-\alpha' & \alpha'+\beta+\gamma & 0 & ;\ z \\ 0 & \alpha'+\beta'+\gamma & \gamma'-\gamma & \end{array}\right\} \\ &= z^{\alpha'}(1-z)^{\gamma}\,P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ 0 & \alpha'+\beta'+\gamma & \gamma'-\gamma & ;\ z \\ \alpha-\alpha' & \alpha'+\beta+\gamma & 0 & \end{array}\right\}. \end{aligned} \tag{4.8}$$

Interchanging the exponent pair γ and γ' will therefore produce further two and hence a total of four different representations together α and α'. On the other hand, mixing each of these four with the six different Möbius transformation relations (4.6) would give a total of twenty four different representations. We shall return to them in a later section.

Let us introduce exponent-differences

$$\lambda := \alpha' - \alpha, \qquad \mu := \beta' - \beta, \qquad \nu := \gamma' - \gamma.$$

Then

$$\alpha + \beta + \gamma = \frac{1}{2}(1 - \lambda - \mu - \nu),$$

and

$$\alpha+\beta'+\gamma=\frac{1}{2}(1-\lambda+\mu-\nu).$$

Thus the (4.7) becomes

$$P\left\{\begin{array}{ccc} 0 & \infty & 1 \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{array} ;\ z\right\} = z^{\alpha}(1-z)^{\gamma}\,P\left\{\begin{array}{ccc} 0 & \infty & 1 \\ 0 & \alpha+\beta+\gamma & 0 \\ \alpha'-\alpha & \alpha+\beta'+\gamma & \gamma'-\gamma \end{array} ;\ z\right\}$$

$$= z^{\alpha}(1-z)^{\gamma}\,P\left\{\begin{array}{ccc} 0 & \infty & 1 \\ 0 & \frac{1}{2}(1-\lambda-\mu-\nu) & 0 \\ \lambda & \frac{1}{2}(1-\lambda+\mu-\nu) & \nu \end{array} ;\ z\right\}. \tag{4.9}$$

Hence the corresponding equation becomes

$$\frac{d^2w}{dz^2}+\Big(\frac{1-\lambda}{z}+\frac{1-\nu}{z-1}\Big)\frac{dw}{dz}+\Big[\frac{(1-\lambda-\mu-\nu)(1-\lambda+\mu-\nu)}{4z(z-1)}\Big]\,w=0. \tag{4.10}$$

A final reduced from is obtained by eliminating the first derivative term from (2.5). It is now regarded as in the **normal form** (or sometimes called one-dimensional Schrödinger form) by selecting $\alpha+\alpha'=1$ and $\gamma+\gamma'=1$ in (4.9). That is,

$$z^{\frac{1}{2}(1-\lambda)}(z-1)^{\frac{1}{2}(1-\nu)}P\left\{\begin{array}{ccc} 0 & \infty & 1 \\ 0 & \frac{1}{2}(1-\lambda-\mu-\nu) & 0 \\ \lambda & \frac{1}{2}(1-\lambda+\mu-\nu) & \nu \end{array} ;\ z\right\}$$

$$= P\left\{\begin{array}{ccc} 0 & \infty & 1 \\ \frac{1}{2}(1-\lambda) & -\frac{1}{2}(1+\mu) & \frac{1}{2}(1-\nu) \\ \frac{1}{2}(1+\lambda) & -\frac{1}{2}(1-\mu) & \frac{1}{2}(1+\nu) \end{array} ;\ z\right\}$$

So the corresponding (2.5) becomes

$$\frac{d^2w}{dz^2}+\frac{1}{4}\Big(\frac{1-\lambda^2}{z^2}+\frac{1-\nu^2}{(z-1)^2}+\frac{\lambda^2-\mu^2+\nu^2-1}{z(z-1)}\Big)\,w=0. \tag{4.11}$$

5. Gauss Hypergeometric Equation

Let us re-derive some of the calculations done in the last section

$$t=M(z)=\frac{(z-\xi)(\eta-\zeta)}{(z-\eta)(\xi-\zeta)}$$

be a Möbius transformation. We deduce from Exercise 4.6 that

$$P\left\{\begin{array}{ccc} \xi & \eta & \zeta \\ \alpha & \beta & \gamma; \\ \alpha' & \beta' & \gamma' \end{array} z\right\} = P\left\{\begin{array}{ccc} 0 & \infty & 1 \\ \alpha & \beta & \gamma; \\ \alpha' & \beta' & \gamma' \end{array} t\right\}$$

$$= t^{\alpha}(1-t)^{\gamma}P\left\{\begin{array}{ccc} 0 & \infty & 1 \\ \alpha-\alpha & \alpha+\beta+\gamma & \gamma-\gamma; \\ \alpha'-\alpha & \alpha+\beta'+\gamma & \gamma'-\gamma \end{array} t\right\}$$

$$= t^{\alpha}(1-t)^{\gamma}P\left\{\begin{array}{ccc} 0 & \infty & 1 \\ 0 & \alpha+\beta+\gamma & 0; \\ \alpha'-\alpha & \alpha+\beta'+\gamma & \gamma'-\gamma \end{array} t\right\}.$$

It turns out that it is more efficient to set the following parametrization of the exponent pairs $\alpha+\beta+\gamma=a$, $\alpha+\beta'+\gamma=b$ and $1-c=\alpha'-\alpha$. Note that $c-a-b=1-\gamma-\gamma'-\beta-\beta'-2\gamma=\gamma+\gamma'-2\gamma=\gamma'-\gamma$. Then the last Riemann-Scheme becomes

$$P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ 0 & a & 0\,; & t \\ 1-c & b & c-a-b & \end{array}\right\}$$

satisfies the differential equation

$$\frac{d^2w}{dt^2}+\Big(\frac{1-(1-c)}{t}+\frac{1-(c-a-b)}{t-1}\Big)\frac{dw}{dt}+\frac{ab}{t(t-1)}\,w=0,$$

or

$$t(1-t)w''+\{c-(a+b+1)t\}w'-abw=0, \tag{5.1}$$

which is called the **Gauss hypergeometric equation**, where we have assumed that none of the exponent pairs $\{0,\,1-c\}$, $\{0,\,c-a-b\}$ and $\{a,\,b\}$ is an integer. Note again that the sum of all exponents is unity.

The important **associated Legendre's differential equation**

$$(1-z^2)\frac{d^2w}{dz^2}-2z\frac{dw}{dz}+\Big\{\nu(\nu+1)-\frac{\mu^2}{1-z^2}\Big\}w=0$$

is a Gauss hypergeometric differential equation, where ν, μ are arbitrary complex constants, with singularities at ±1, ∞. In terms of Riemann's P notation, we have

$$w(z)=P\left\{\begin{array}{cccc} -1 & \infty & 1 & \\ \frac{\mu}{2} & \nu+1 & \frac{\mu}{2}\,; & t \\ -\frac{\mu}{2} & \nu & -\frac{\mu}{2} & \end{array}\right\}.$$

The equation is called **Legendre's differential equation** if $m=0$. Its solutions are called **spherical harmonics**. This equation is important because it is often encountered in solving boundary value problems of potential theory for spherical regions, and hence the name of spherical harmonics.

5.1. **Relation with confluent hypergeometric equation.** A transformation of $Y(z)=y(z/b)$ in the (5.1) yields

$$z(1-z/b)\frac{d^2Y}{dz^2}+[c-(a+b+1)z/b]\frac{dY}{dz}-a\,Y=0 \tag{5.2}$$

which clearly has regular singular points at 0, b, ∞. Letting $b\to\infty$ in the equation (5.2) results in the

$$z\frac{d^2y}{dz^2}+(c-z)\frac{dy}{dz}-a\,y=0, \tag{5.3}$$

which is called the **confluent hypergeometric equation**.

5.2. **Relation with Bessel Equations.** We define the **Bessel function of first kind of order** ν to be the complex function represented by the power series

$$J_\nu(z) = \sum_{k=0}^{+\infty} \frac{(-1)^k (\frac{1}{2}z)^{\nu+2k}}{\Gamma(\nu+k+1)\,k!} = z^\nu \sum_{k=0}^{+\infty} \frac{(-1)^k (\frac{1}{2})^{\nu+2k}}{\Gamma(\nu+k+1)\,k!}\, z^{2k}. \tag{5.4}$$

Here ν is an arbitrary complex constant.

Let us set $a = \nu + \frac{1}{2}$, $c = 2\nu + 1$ and replace z by $2iz$ in the Kummer series. That is,

$$\begin{aligned}
\Phi(\nu+1/2,\, 2\nu+1; 2iz) &= \sum_{k=0}^{\infty} \frac{(\nu+\frac{1}{2})_k}{(2\nu+1)_k\, k!} (2iz)^k \\
&= {}_1F_1\left(\begin{matrix} \nu+\frac{1}{2} \\ 2\nu+1 \end{matrix} \middle| \, 2iz \right) \\
&= e^{iz}\, {}_0F_1\left(\begin{matrix} - \\ \nu+1 \end{matrix} \middle| \, -\big(\frac{z}{2}\big)^2 \right) \\
&= e^{iz} \sum_{k=0}^{\infty} \frac{(-1)^k}{(\nu+1)_k\, k!} \frac{z^{2k}}{2^{2k}} \\
&= e^{iz} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\nu+1)}{\Gamma(\nu+k+1)\, k!} \big(\frac{z}{2}\big)^{2k} \\
&= e^{iz} \Gamma(\nu+1) \big(\frac{z}{2}\big)^{-\nu} \cdot \big(\frac{z}{2}\big)^{\nu} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\nu+k+1)\, k!} \big(\frac{z}{2}\big)^{2k} \\
&= e^{iz} \Gamma(\nu+1) \big(\frac{z}{2}\big)^{-\nu} \cdot J_\nu(z),
\end{aligned}$$

where we have applied Kummer's second transformation

$${}_1F_1\left(\begin{matrix} a \\ 2a \end{matrix} \middle| \, 4x \right) = e^{2x}\, {}_0F_1\left(\begin{matrix} - \\ a+\frac{1}{2} \end{matrix} \middle| \, x^2 \right) \tag{5.5}$$

in the third step above, and the identity $(\nu)_k = \Gamma(\nu+k)/\Gamma(\nu)$ in the fourth step.

Since the confluent hypergeometric function $\Phi(\nu+1/2,\, 2\nu+1; 2iz)$ satisfies the confluent hypergeometric equation

$$z\frac{d^2y}{dz^2} + (c-z)\frac{dy}{dz} - a\,y = 0 \tag{5.6}$$

with $a = \nu + 1/2$ and $c = 2\nu + 1$. Substituting $e^{iz}\Gamma(\nu+1)\big(\frac{z}{2}\big)^{-\nu} \cdot J_\nu(z)$ into the (5.6) and simplfying lead to the equation

$$z^2\frac{d^2y}{dz^2} + x\frac{dy}{dz} + (z^2 - \nu^2)y = 0. \tag{5.7}$$

This is called the **Bessel equation with parameter** ν. It is clearly seen that $z = 0$ is a regular singular point and that $Z = \frac{1}{z}$ has an irregular singular point at $Z = \frac{1}{0} = \infty$ for the transformed equation with $Y(z) = y(1/z)$.

The above derivation shows that the Bessel function of the first kind with order ν is a special case of the confluent hypergeometric functions with specifically chosen parameters.

5.3. **Relation with Hermite equation.** To be completed.

6. Gauss Hypergeometric series

Let us consider power series solutions of the Gauss Hypergeometric equation

$$z(1-z)w'' + \{c-(a+b+1)z\}w' - abw = 0, \tag{6.1}$$

at the regulary singular point $z=0$. Substitute the power series $w = \sum_{k=0}^{\infty} a_k z^{k+\alpha}$ into the equation (6.1) yields the indicial equation

$$\alpha(\alpha+c-1)=0,$$

so that the coefficient a_k satisfies the recurrence relation

$$(\alpha+k)(\alpha+c+k-1)a_k = (\alpha+a+k-1)(\alpha+b+k-1)a_{k-1},$$

or

$$(\alpha+1+k-1)(\alpha+c+k-1)a_k = (\alpha+a+k-1)(\alpha+b+k-1)a_{k-1}.$$

This gives

$$\begin{aligned}
a_k &= \frac{(\alpha+a+k-1)(\alpha+b+k-1)}{(\alpha+1+k-1)(\alpha+c+k-1)}a_{k-1} \\
&= \frac{(\alpha+a+k-1)(\alpha+b+k-1)}{(\alpha+1+k-1)(\alpha+c+k-1)}\frac{(\alpha+a+k-2)(\alpha+b+k-2)}{(\alpha+1+k-2)(\alpha+c+k-2)}a_{k-2} \\
&\cdots\cdots \\
&= \frac{(\alpha+a+k-1)(\alpha+b+k-1)}{(\alpha+1+k-1)(\alpha+c+k-1)}\cdots\frac{(\alpha+a+0)(\alpha+b+0)}{(\alpha+1+0)(\alpha+c+0)}a_0 \\
&= \frac{(\alpha+a)_k(\alpha+b)_k}{(\alpha+1)_k(\alpha+c)_k}a_0,
\end{aligned}$$

where a_0 is an arbitrarily chosen constant and we recall the Pochhammer notation that

$$(\alpha)_0 = 1, \qquad (\alpha)_k = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1),$$

for each integer k. So we have obtained, assuming that $1-c$ is not an integer, two linearly independent power series solutions

$$w_1(z) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{k!(c)_k}z^k, \qquad w_2(z) = z^{1-c}\sum_{k=0}^{\infty}\frac{(a-c+1)_k(b-c+1)_k}{(2-c)_k k!}z^k.$$

Definition 6.1. A **Gauss hypergeometric series** is a power series given in the form

$$\begin{aligned}
F(a,\,b;\,c;\,z) = {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}\middle|\, z\right) &= \sum_{k=0}^{\infty} c_k z^k = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k\,k!}z^k \\
&= 1 + \frac{ab}{c}\,z + \frac{a(a+1)b(b+1)}{c(c+1)\,1!}\,z^2 + \\
&\quad + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)\,2!}\,z^3 + \cdots
\end{aligned} \tag{6.2}$$

We note that the Gauss series above terminates into a polynomial when either a or b is zero or a negative integer. The second series terminates when either $a-c+1$ or $b-c+1$ is zero or a negative integer. These polynomials are called Jacobi polynomials. its limiting cases include (orthogonal) Laguerre polynomials and Hermite polynomials. See [1]

We note that the hypergeometric function includes many important functions that we have encountered. For examples, we have

$$\begin{aligned}
(1-z)^{-a} &= {}_1F_0\left(\begin{matrix} a \\ - \end{matrix}\middle|\, -z\right);\\
e^z &= z\,{}_0F_0\left(\begin{matrix} - \\ - \end{matrix}\middle|\, z\right);\\
\log(1+z) &= z\,{}_2F_1\left(\begin{matrix} 1,1 \\ 2 \end{matrix}\middle|\, -z\right);\\
\sin z &= z\,{}_0F_1\left(\begin{matrix} - \\ \frac{3}{2} \end{matrix}\middle|\, -\frac{z^2}{4}\right);\\
\cos z &= z\,{}_0F_1\left(\begin{matrix} - \\ \frac{1}{2} \end{matrix}\middle|\, -\frac{z^2}{4}\right);\\
\sin^{-1} z &= z\,{}_2F_1\left(\begin{matrix} \frac{1}{2},\frac{1}{2} \\ \frac{3}{2} \end{matrix}\middle|\, z^2\right)
\end{aligned}$$

The series derives its name because the ratio between two consecutive terms, that is $\dfrac{a_{k+1}}{a_k}$ is a *rational function* of k.

Theorem 6.1 (Chu (1303)-Vandermonde (1772))**.** *Let n be a positive integer. Then*

$$ {}_2F_1\left(\begin{matrix} -n, b \\ c \end{matrix}\middle|\, 1\right) = \frac{(c-b)_n}{(c)_n}. \tag{6.3}$$

Remark 6.2. According to Askey's account [1], the Chinese mathematician S.-S. Chu [2] had already discovered the essential form of the identity four centuries before the French mathematician Vandermonde. Askey described Chu's result was "absolutely incredible" as it was discovered even before adequate mathematical notation was available [1, p. 60].

Theorem 6.3. *The series ${}_2F_1(a,\, b;\, c;\, z)$*

(1) *converges absolutely in $|z|<1$;*
(2) *converges absolutely if $\Re(c-a-b)>0$ if $|z|=1$;*
(3) *diverges if $\Re(c-a-b)\le -1$.*

Proof. (1) This first part is easy. For let $u_k = \dfrac{(a)_k(b)_k}{(c)_k k!} z^k$. Then

$$\left|\frac{u_{k+1}}{u_k}\right| = \left|\frac{(a+k)(b+k)}{(c+k)(1+k)} z\right| \to |z|$$

as $k\to\infty$. Thus the hypergeometric series converges (uniformly) within the unit disc, and it diverges in $|z|>1$ by the ratio test.

(2) Recall that
$$\Gamma(z) = \lim_{k\to\infty} k!k^{z-1}/(z)_k.$$
But then the $k-$th coefficient of the power series behaves asymptotically like
$$\frac{(a)_k(b)_k}{(c)_k k!} \approx \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{k!k^{a-1}\cdot k!k^{b-1}}{k!k^{c-1}\cdot k!} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\, k^{a+b-c-1}$$
$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{1}{k^{c+1-a-b}},$$
for all $n \geq N$ say. But then
$$\left|\sum_N^\infty \frac{(a)_k(b)_k}{(c)_k k!} z^k\right| \leq \left|\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\right| \sum_N^\infty \frac{1}{|k^{c+1-a-b}|} = \left|\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\right| \sum_N^\infty \frac{1}{k^{\Re(c+1-a-b)}}$$
which is known to be convergent when $\Re(c-a-b) > 0$ since $\sum \frac{1}{k^{1+\delta}}$ converges when $\Re\delta > 0$. Alternatively, we observe that
$$\frac{c_k}{c_{k+1}} = \frac{(k+1)(c+k)}{(a+k)(b+k)} = 1 + \frac{1+c-a-b}{k} + O\big(\frac{1}{k^2}\big),$$
that is,
$$\left|\frac{c_k}{c_{k+1}}\right| \geq 1 + \frac{1+\Re(c-a-b)}{k} + O\big(\frac{1}{k^2}\big).$$
Hence Raabe's test ($\underline{\lim}_{k\to\infty} k(|\frac{c_k}{c_{k+1}}| - 1) > 1$ means convergent, and $\overline{\lim}_{k\to\infty} k(|\frac{c_k}{c_{k+1}}| - 1) < 1$ implies divergent) implies that the Gauss series converges on $|z| = 1$, $z \neq 1$.

(3) In this case, since the general term of the series does not tend to zero (please verify), so the series diverges.

□

We are ready for

Theorem 6.4 (Gauss 1812)**.** *Suppose* $\Re(c-a-b) > 0$. *Then*
$$\sum_{k=0}^\infty \frac{(a)_k(b)_k}{(c)_k\, k!} = {}_2F_1\Big(\begin{matrix} a, b \\ c \end{matrix}\Big|\, 1\Big) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$
holds.

Proof. Suppose that $\Re(c-a-b) > 0$. Then Theorem 6.3 implies that $F(a,\, b;\, c;\, 1)$ converges. Abel's theorem asserts that
$$F(a,\, b;\, c;\, 1) = \lim_{x\to 1-} F(a,\, b;\, c;\, x).$$
Let us write,
$$F(a,\, b;\, c;\, z) = \sum_{k=0}^\infty A_k z^k, \quad F(a,\, b;\, c+1;\, z) = \sum_{k=0}^\infty B_k z^k.$$
Then it is routine to check that
$$c(c-a-b)A_k - (c-a)(c-b)B_k = \frac{(a)_k(b)_k}{k!(c+1)_{k-1}}\Big[c-a-b-\frac{(c-a)(c-b)}{c+k}\Big]$$

and

$$c(kA_k - (k+1)A_{k+1}) = \frac{(a)_k(b)_k}{k!(c+1)_{k-1}}\Big[k - \frac{(a+k)(b+k)}{c+k}\Big]$$

hold and that the right-hand sides of the above identities are equal. Thus, we deduce

$$c(c-a-b)A_k = (c-a)(c-b)B_k + ckA_k - c(k+1)A_{k+1}.$$

Hence

$$c(c-a-b)\sum_{k=0}^{n} A_k = (c-a)(c-b)\sum_{k=0}^{n} B_k - c(n+1)A_{n+1}.$$

Now let $n \to \infty$ to establish

$$F(a,\, b;\, c;\, 1) = \frac{(c-a)(c-b)}{c(c-a-b)}F(a,\, b;\, c+1;\, 1) \tag{6.4}$$

since

$$(n+1)A_{n+1} \approx \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\frac{1}{(n+1)^{c-a-b}} \to 0$$

and $\Re(c-b-a) > 0$. Iterating (6.4) $k-1$ times yields

$$F(a,\, b;\, c;\, 1) = \frac{(c-a)_k(c-b)_k}{(c)_k(c-a-b)_k}F(a,\, b;\, c+k;\, 1). \tag{6.5}$$

Recall that

$$\begin{aligned}\Gamma(a+k) &= (a+k)\Gamma(a+k)\\ &= (a+k)(a+k-1)\Gamma(a+k-1)\\ &= (a+k)(a+k-1)\cdots(a+k-k)\cdots\Gamma(a+k-k)\\ &= (a+k)(a+k-1)\cdots(a+1)(a)\Gamma(a)\\ &= (a)_k\Gamma(a)\end{aligned}$$

or

$$\Gamma(a+k) = (a)_k\Gamma(a).$$

Applying this to (6.5) yields

$$\frac{\Gamma(c-a)\Gamma(c-b)}{\Gamma(c)\Gamma(c-a-b)}F(a,\, b;\, c;\, 1) = \frac{\Gamma(c+k-a)\Gamma(c+k-b)}{\Gamma(c+k)\Gamma(c+k-a-b)}F(a,\, b;\, c+k;\, 1).$$

We deduce from Stirling's formula

$$\Gamma(z) = \sqrt{2\pi}\, e^{-z} z^{z-\frac{1}{2}}\left(1 + o(\frac{1}{z})\right)$$

that

$$\frac{\Gamma(c+k-a)\Gamma(c+k-b)}{\Gamma(c+k)\Gamma(c+k-a-b)} \to 1, \quad k \to \infty.$$

It remains to prove the limit

$$\lim_{k\to\infty}\frac{\Gamma(c-a)\Gamma(c-b)}{\Gamma(c)\Gamma(c-a-b)}F(a,\, b;\, c;\, 1) = \lim_{k\to\infty} F(a,\, b;\, c+k;\, 1) = 1 \tag{6.6}$$

under the assumption that $\Re(c-b-a) > 0$. □

Exercise 6.5. Complete the proof for (6.6). See [3].

6.1. **Euler's transformation.** The above constructed a solution in $|z| < 1$ (and also $|z| = 1$ with $\Re(c-a-b) > 0$). The ${}_2F_1$ is more "global" than that.

Theorem 6.6 (Euler)**.** *Let* $\Re c > \Re b > 0$. *Then*

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| \, z \right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt \tag{6.7} $$

where we assume the branch to be $[1, +\infty)$ *and* $\arg t = \arg(1-t) = 0$.

This allows us to analytic continue the hypergeometric functions from $|z| < 1$ to $\mathbb{C}\backslash[1, +\infty)$.

The Euler transformation above can be re-interpreted as *Riemann-Liouville fractional derivative.*

7. Symmetry: Kummer's 24 Solutions

We now consider *internal symmetry* of the hypergeometric equation/function. Let us follow Riemann's approach. We shall write down the pairs of power series of the fundamential sets of solutions with respect to the three regular singular points $\{0, 1, \infty\}$. Riemann uses the notation $\{P^{(\alpha)}, P^{(\alpha')}\}$, $\{P^{(\beta)}, P^{(\beta')}\}$, $\{P^{(\gamma)}, P^{(\gamma')}\}$, while we shall use $\{w_1(z), w_2(z)\}$, $\{w_3(z), w_4(z)\}$, $\{w_5(z), w_6(z)\}$ instead.

Recall that the standard hypergeometric equation (6.1) has two power series solutions whose characteristic exponents 0 and $1-c$ are recorded in the Riemann-Scheme

$$ P\left\{ \begin{matrix} 0 & \infty & 1 & \\ 0 & a & 0\,; & z \\ 1-c & b & c-a-b & \end{matrix} \right\} $$

We have the following observation

$$ \begin{aligned} P\left\{ \begin{matrix} 0 & \infty & 1 & \\ 0 & a & 0\,; & z \\ \underline{1-c} & b & c-a-b & \end{matrix} \right\} &= z^{1-c} P\left\{ \begin{matrix} 0 & \infty & 1 & \\ c-1 & a-c+1 & 0\,; & z \\ \underline{0} & b-c+1 & c-a-b & \end{matrix} \right\} \\ &= z^{1-c} P\left\{ \begin{matrix} 0 & \infty & 1 & \\ 1-c' & a' & 0\,; & z \\ \underline{0} & b' & c'-a'-b' & \end{matrix} \right\}, \end{aligned} $$

where $a' = a-c+1$, $b' = b-c+1$ and $c' = 2-c$. The above transformation indicates that one of the original power series solution of equation (6.1) with characteristic exponent $1-c$ can be written in terms of the characteristic exponent 0 power series solution $F(a', b'; c'; z)$ represented by the above last Riemann-Scheme. Thus we have two solutions

$$ w_1(z) = {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| \, z \right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!(c)_k} z^k, $$

$$ w_2(z) = z^{1-c}\, {}_2F_1\left(\begin{matrix} a-c+1, b-c+1 \\ 2-c \end{matrix} \middle| \, z \right) = z^{1-c} \sum_{k=0}^{\infty} \frac{(a-c+1)_k (b-c+1)_k}{(2-c)_k k!} z^k, $$

which are precisely two linearly independent solutions found in Section 6. We recall that we have assumed that $1-c$ is not an integer. Otherwise, there will be a logarithm term in the solution (see [9]). This gives two series solutions around the regular singular point $z = 0$.

We now consider the regular singular point $z = 1$. We now consider the transformation $z = 1 - t$. This will transform the hypergeometric equation to the form

$$t(1-t)\frac{d^2w(t)}{dt^2} + [(1+a+b-c) - (a+b+1)t]\frac{dw(t)}{dt} - abw(t) = 0,$$

where $w(t) = y(z)$ where $t = 1 - z$. That is, we have shown that the relation

$$P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ 0 & a & 0\,; & z \\ 1-c & b & c-a-b & \end{array}\right\} = P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ 0 & a & 0\,; & 1-z \\ c-a-b & b & 1-c & \end{array}\right\} \tag{7.1}$$

holds. Observe that

$$\begin{aligned} P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ 0 & a & \underline{0} & ;\ z \\ 1-c & b & c-a-b & \end{array}\right\} &= P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ \underline{0} & a & 0 & ;\ 1-z \\ c-a-b & b & 1-c & \end{array}\right\} \\ &= P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ \underline{0} & a' & 0 & ;\ 1-z \\ 1-c' & b' & c'-a'-b' & \end{array}\right\} \end{aligned}$$

where $a' = a,\ b' = b$ and $c' = 1 + a + b - c$. We obtain

$$w_3(z) = {}_2F_1\Big(\begin{array}{c} a,\ b \\ 1+a+b-c \end{array}\Big|\ 1-z\Big).$$

Applying the transformation from w_1 to w_2 above to w_3 and identifying $t = 1 - z$ and the parameters a', b' and c' yields the fourth solution

$$\begin{aligned} w_4(z) &= t^{1-c'}\,{}_2F_1\Big(\begin{array}{c} a'-c'+1,\ b'-c'+1 \\ 2-c' \end{array}\Big|\ t\Big) \\ &= (1-z)^{c-a-b}\,{}_2F_1\Big(\begin{array}{c} c-b,\ c-a \\ 1+c-a-b \end{array}\Big|\ 1-z\Big). \end{aligned}$$

Similarly, one can obtain two solutions around the regular singular point ∞:

Exercise 7.1.

$$w_5(z) = (-z)^{-a}\,{}_2F_1\Big(\begin{array}{c} a,\ a-c+1 \\ a-b+1 \end{array}\Big|\ \frac{1}{z}\Big),$$

where $c - a - b$ is not an integer, and

$$w_6(z) = (-z)^{-b}\,{}_2F_1\Big(\begin{array}{c} b,\ b-c+1 \\ b-a+1 \end{array}\Big|\ \frac{1}{z}\Big),$$

where $a - b$ is not an integer. The negative signs are introduced for convinence sake (see later).

Hence we have obtained three pairs of six solutions $w_1(z), \cdots, w_6(z)$ in total. Let us now take $w_1(z)$, that is the standard solution, as an example on how to generate more solutions (where we have used underline on the appropriate exponent

to indicate on which solution that we are considering):

$$\begin{aligned} w_1(z) &= {}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix}\middle|\, z\right) \\ &= P\left\{\begin{matrix} 0 & \infty & 1 & \\ \underline{0} & a & 0\,; & z \\ 1-c & b & c-a-b & \end{matrix}\right\} \\ &= (1-z)^{c-a-b} P\left\{\begin{matrix} 0 & \infty & 1 & \\ \underline{0} & c-b & a+b-c\,; & z \\ 1-c & c-a & 0 & \end{matrix}\right\}. \end{aligned}$$

Thus

$${}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix}\middle|\, z\right) = C(1-z)^{c-a-b}\,{}_2F_1\left(\begin{matrix} c-a,c-b \\ c \end{matrix}\middle|\, z\right),$$

for some constant C. If we choose the branch of $(1-z)^{c-a-b}$ such that it equals 1 when $z=0$, then $C=1$ since both sides of the equation are equal to 1. Thus, we have

$$w_1(z) = {}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix}\middle|\, z\right) = (1-z)^{c-a-b}\,{}_2F_1\left(\begin{matrix} c-a,c-b \\ c \end{matrix}\middle|\, z\right),$$

where $|\arg(1-z)| < \pi$. On the other hand, we have

$$\begin{aligned} P\left\{\begin{matrix} 0 & \infty & 1 & \\ \underline{0} & a & 0\,; & z \\ 1-c & b & c-a-b & \end{matrix}\right\} &= P\left\{\begin{matrix} 0 & \infty & 1 & \\ \underline{0} & 0 & a\,; & \dfrac{z}{z-1} \\ 1-c & c-a-b & b & \end{matrix}\right\} \\ &= \left(1-\frac{z}{z-1}\right)^a P\left\{\begin{matrix} 0 & \infty & 1 & \\ \underline{0} & a & 0\,; & \dfrac{z}{z-1} \\ 1-c & c-b & b-a & \end{matrix}\right\} \\ &= (1-z)^{-a} P\left\{\begin{matrix} 0 & \infty & 1 & \\ \underline{0} & a & 0\,; & \dfrac{z}{z-1} \\ 1-c & c-b & b-a & \end{matrix}\right\}. \end{aligned}$$

Thus, we have

$${}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}\middle|\, z\right) = (1-z)^{-a}\,{}_2F_1\left(\begin{matrix} a,\ c-b \\ c \end{matrix}\middle|\, \frac{z}{z-1}\right),$$

provided that $|\arg(1-z)| < \pi$. Interchanging the a and b in the above analysis yields another format:

$${}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}\middle|\, z\right) = (1-z)^{-b}\,{}_2F_1\left(\begin{matrix} b,\ c-a \\ c \end{matrix}\middle|\, \frac{z}{z-1}\right),$$

provided that $|\arg(1-z)| < \pi$. Putting these cases together, we have

$$\begin{aligned} w_1(z) &= {}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix} \middle| \, z \right) \\ &= (1-z)^{c-a-b}{}_2F_1\left(\begin{matrix} c-a, c-b \\ c \end{matrix} \middle| \, z \right) \\ &= (1-z)^{-a}{}_2F_1\left(\begin{matrix} a,\ c-b \\ c \end{matrix} \middle| \, \frac{z}{z-1} \right) \\ &= (1-z)^{-b}{}_2F_1\left(\begin{matrix} b,\ c-a \\ c \end{matrix} \middle| \, \frac{z}{z-1} \right). \end{aligned}$$

Similar formulae exist for the remaining $w_2(z), \cdots, w_6(z)$, where each solution has three additional variartions, thus forming a total of twenty four solutions, which are called **Kummer's 24** solutions (Crelle, issue XV (1836)). We list the remaining sets of these formulae below.

Exercise 7.2.

$$\begin{aligned} w_2(z) &= z^{1-c}{}_2F_1(a+1-c,\, b+1-c;\, 2-c;\, z) \\ &= z^{1-c}(1-z)^{c-a-b}{}_2F_1(1-a,\, 1-b;\, 2-c;\, z) \\ &= z^{1-c}(1-z)^{c-a-1}{}_2F_1(a+1-c,\, 1-b;\, 2-c;\, z/(z-1)) \\ &= z^{1-c}(1-z)^{c-b-1}{}_2F_1(b+1-c,\, 1-a;\, 2-c;\, z/(z-1)), \end{aligned}$$

Exercise 7.3.

$$\begin{aligned} w_3(z) =& {}_2F_1(a,\, b;\, a+b-c+1;\, 1-z) \\ &= z^{1-c}{}_2F_1(a-c+1,\, b-c+1;\, a+b-c+1;\, 1-z) \\ &= z^{-a}{}_2F_1(a,\, a-c+1;\, a+b-c+1;\, (z-1)/z) \\ &= z^{-b}{}_2F_1(b,\, b-c+1;\, a+b-c+1;\, (z-1)/z), \end{aligned}$$

Exercise 7.4.

$$\begin{aligned} w_4(z) =& (1-z)^{c-a-b}{}_2F_1(c-a,\, c-b;\, c-a-b+1;\, 1-z) \\ &= z^{1-c}(1-z)^{c-a-b}{}_2F_1(1-a,\, 1-b;\, c-a-b+1;\, 1-z) \\ &= z^{-a}(1-z)^{c-a-b}{}_2F_1(1-a,\, c-a;\, c-a-b+1;\, (z-1)/z) \\ &= z^{-b}(1-z)^{c-a-b}{}_2F_1(1-b,\, c-b;\, c-a-b+1;\, (z-1)/z), \end{aligned}$$

Exercise 7.5.

$$\begin{aligned} w_5(z) &= (-z)^{-a}{}_2F_1(a,\, a-c+1;\, a-b+1;\, 1/z) \\ &= (-z)^{-a}(1-1/z)^{c-a-b}{}_2F_1(1-b,\, c-b;\, a+1-b;\, 1/z) \\ &= (-z)^{-a}(1-1/z)^{-a}{}_2F_1(a,\, c-b;\, a+1-b;\, 1/(1-z)) \\ &= (-z)^{-a}(1-1/z)^{c-a-1}{}_2F_1(a+c-1,\, 1-b;\, a+1-b;\, 1/(1-z)) \end{aligned}$$

and

Exercise 7.6.

$$\begin{aligned} w_6(z) &= (-z)^{-b}{}_2F_1(b,\, b-c+1;\, b-a+1;\, 1/z) \\ &= (-z)^{-b}(1-1/z)^{c-a-b}{}_2F_1(1-a,\, c-a;\, b+1-a;\, 1/z) \\ &= (-z)^{-b}(1-1/z)^{-b}{}_2F_1(b,\, c-a;\, b+1-a;\, 1/(1-z)) \\ &= (-z)^{-b}(1-1/z)^{c-b-1}{}_2F_1(b-c+1,\, 1-a;\, b+1-a;\, 1/(1-z)). \end{aligned}$$

One can also interpret the above twenty four solutions by considering that there are six Möbius transformations that permute the three regular singular points $\{0,\, 1,\, \infty\}$:

$$z, \qquad 1-z, \qquad \frac{z}{z-1}, \qquad \frac{1}{z}, \qquad \frac{1}{z-1}, \qquad 1-\frac{1}{z}.$$

This list gives all the possible permutations of the regular singularities. Notice the function

$$z^{\rho}(1-z)^{\sigma}{}_2F_1(a^*,\, b^*;\, c^*;\, z^*),$$

where ρ and σ can be chosen so that exactly one of the exponents at $z=0$ and $z=1$ becomes zero. But we have two choices of exponents at either of the points (0 or 1) so that there are a total of $6\times 2\times 2 = 24$ combinations. We refer to Bateman's project Vol. I [4] for a complete list of these twenty four solutions.

8. Riemann's monodromy group

Recall the notation Riemann used

$$P^{(\alpha)},\ P^{(\alpha')};\ P^{(\beta)},\ P^{(\beta')};\ P^{(\gamma)},\ P^{(\gamma')}$$

to denote two branches of the regular singularities at 0, 1, ∞ respectively. Then the following figure shows that three appropriate solutions from the Kummer list of 24 solutions would satisfy the following equations

$$P^{(\alpha)} = \alpha_\beta P^{(\beta)} + \alpha_{\beta'} P^{(\beta')} = \alpha_\gamma P^{(\gamma)} + \alpha_{\gamma'} P^{(\gamma')}, \tag{8.1}$$

and

$$P^{(\alpha')} = \alpha'_\beta P^{(\beta)} + \alpha'_{\beta'} P^{(\beta')} = \alpha'_\gamma P^{(\gamma)} + \alpha'_{\gamma'} P^{(\gamma')} \tag{8.2}$$

in the region as shown below:

The problem that Riemann tackled was that he showed that the ratios

$$\frac{\alpha_\beta}{\alpha'_\beta},\ \frac{\alpha'_\beta}{\alpha'_{\beta'}},\ \frac{\alpha_\gamma}{\alpha'_\gamma},\ \frac{\alpha_{\gamma'}}{\alpha'_{\gamma'}}$$

are *invariants* with respect to the constant multiplier for each branch chosen.

We now take a simple closed curve with positive orientation enclosing the points $x = 0$ and $x = 1$ beginning and ending at a point in the upper half-plane. On the one hand, this is equivalent to a negatively oriented curve around the infinity ∞. One the other hand, the analytic continuation principle implies that the combined effect of the simple curve surrounding $x = 0$ and $x = 1$ is effected by the sequence

$$P^{(\alpha)} \mapsto\ e^{\alpha 2\pi i} P^{(\alpha)} \mapsto\ e^{2\pi i\alpha}\big(\alpha_\gamma e^{2\pi i\gamma} P^{(\gamma)} + \alpha_{\gamma'} e^{2\pi i\gamma'} P^{(\gamma')}\big)$$

Combining with the simultaneous negatively oriented curve about ∞, we deduce

$$e^{-2\pi i\beta}\alpha_\beta P^{(\beta)} + e^{-2\pi i\beta'}\alpha_{\beta'} P^{(\beta')} = e^{2\pi i\alpha}\big(\alpha_\gamma e^{2\pi i\gamma} P^{(\gamma)} + \alpha_{\gamma'} e^{2\pi i\gamma'} P^{(\gamma')}\big) \tag{8.3}$$

$$\alpha'_\beta e^{-2\pi i\beta} P^{(\beta)} + \alpha'_{\beta'} e^{-2\pi i\beta'} P^{(\beta')} = e^{2\pi i\alpha'}\big(\alpha'_\gamma e^{2\pi i\gamma} P^{(\gamma)} + \alpha'_{\gamma'} e^{2\pi i\gamma'} P^{(\gamma')}\big) \tag{8.4}$$

Substitute (8.1) (the second equality) into (8.3) yields

$$\begin{aligned} &\alpha_\beta e^{-2\pi i\beta} P^{(\beta)} + e^{-2\pi i\beta'}\big(\alpha_\gamma P^{(\gamma)} + \alpha_{\gamma'} P^{(\gamma')} - \gamma_\beta P^{(\beta)}\big) \\ &\qquad\qquad = e^{2\pi i\alpha}\big(\alpha_\gamma e^{2\pi i\gamma} P^{(\gamma)} + \alpha_{\gamma'} e^{2\pi i\gamma'} P^{(\gamma')}\big) \end{aligned} \tag{8.5}$$

Collecting like terms yields

$$\begin{aligned}\alpha_\beta P^{(\beta)}(e^{-2\pi i\beta}-e^{-2\pi i\beta'}) &= \alpha_\gamma(e^{(\alpha+\gamma)2\pi i}-e^{-2\pi i\beta'})P^{(\gamma)}\\ &\quad+\alpha_{\gamma'}(e^{(\alpha+\gamma')2\pi i}-e^{-2\pi i\beta'})P^{(\gamma')}\end{aligned}\tag{8.6}$$

Multiplying $e^{\pi(\beta+\beta')}$ on both sides of the above equation and simplifying yields

$$\begin{aligned}\alpha_\beta \sin\pi(\beta'-\beta)P^{(\beta)} &= \alpha_\gamma e^{i\pi(\alpha+\beta+\gamma)}\sin(\alpha+\beta'+\gamma)\pi P^{(\gamma)}\\ &\quad+\alpha_{\gamma'}e^{i\pi(\alpha+\beta+\gamma')}\sin(\alpha+\beta'+\gamma')\pi P^{(\gamma')}.\end{aligned}\tag{8.7}$$

Repeating the above steps for the equations (8.2) and (8.4) yield a similar equation

$$\begin{aligned}\alpha'_\beta \sin\pi(\beta'-\beta)P^{(\beta)} &= \alpha'_\gamma e^{i\pi(\alpha'+\beta+\gamma)}\sin(\alpha'+\beta'+\gamma)\pi P^{(\gamma)}\\ &\quad+\alpha'_{\gamma'}e^{i\pi(\alpha'+\beta+\gamma')}\sin(\alpha'+\beta'+\gamma')\pi P^{(\gamma')}.\end{aligned}\tag{8.8}$$

An application of $\alpha+\alpha'+\beta+\beta'+\gamma+\gamma'=1$ yields

$$\frac{\alpha_\beta}{\alpha'_\beta}=\frac{\alpha_\gamma e^{i\pi\alpha}\sin\pi(\alpha+\beta'+\gamma)}{\alpha'_\gamma e^{i\pi\alpha'}\sin\pi(\alpha'+\beta'+\gamma)}=\frac{\alpha_{\gamma'}e^{i\pi\alpha}\sin\pi(\alpha+\beta'+\gamma')}{\alpha'_{\gamma'}e^{i\pi\alpha'}\sin\pi(\alpha'+\beta'+\gamma')}$$

We may repeat a similar procedure to obtain two equations for $P^{(\beta')}$ as follow

$$\begin{aligned}\alpha'_\beta \sin\pi(\beta-\beta')P^{(\beta')} &= \alpha_\gamma e^{i\pi(\alpha+\beta'+\gamma)}\sin(\alpha+\beta+\gamma)\pi P^{(\gamma)}\\ &\quad+\alpha_{\gamma'}e^{i\pi(\alpha+\beta'+\gamma')}\sin(\alpha+\beta+\gamma')\pi P^{(\gamma')}.\end{aligned}\tag{8.9}$$

and

$$\begin{aligned}\alpha'_{\beta'} \sin\pi(\beta-\beta')P^{(\beta')} &= \alpha'_\gamma e^{i\pi(\alpha'+\beta'+\gamma)}\sin(\alpha'+\beta+\gamma)\pi P^{(\gamma)}\\ &\quad+\alpha'_{\gamma'}e^{i\pi(\alpha'+\beta'+\gamma')}\sin(\alpha'+\beta+\gamma')\pi P^{(\gamma')}\end{aligned}\tag{8.10}$$

from which we deduce

$$\frac{\alpha_{\beta'}}{\alpha'_{\beta'}}=\frac{\alpha_\gamma e^{i\pi\alpha}\sin\pi(\alpha+\beta+\gamma)}{\alpha'_\gamma e^{i\pi\alpha'}\sin\pi(\alpha'+\beta+\gamma)}=\frac{\alpha_{\gamma'}e^{i\pi\alpha}\sin\pi(\alpha+\beta+\gamma')}{\alpha'_{\gamma'}e^{i\pi\alpha'}\sin\pi(\alpha'+\beta+\gamma')}.$$

8.1. **Evaluation of the invariants.** We draw a cut on the whole of $\mathbb{R}$ and assume the followings:

(1) $P^{(\alpha)}\to 1$, $P^{(\alpha')}\sim x^{1-c}$, as $x\to 0$.
(2) $P^{(\beta)}\sim(1-x)^{-a}$, $P^{(\beta')}\sim(1-x)^{-b}$, $x\to\infty$
(3) $P^{(\gamma)}\to 1$, $P^{(\gamma')}\sim(1-x)^{c-a-b}$, as $x\to 1$.
(4) $0<\arg(x)<\pi$, $-\pi<\arg(1-x)\le 0$.

Recall the following evaluation of Gauss:

$${}_2F_1(a,\,b;\,c;\,1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}:=\phi(x).$$

Consider

$$P^{(\alpha)}=\alpha_\gamma P^{(\gamma)}+\alpha_{\gamma'}P^{(\gamma')},\quad P^{(\alpha')}=\alpha'_\gamma P^{(\gamma)}+\alpha'_{\gamma'}P^{(\gamma')}\tag{8.11}$$

again. We assume $\Re(c-a-b)>0$ and applying the first of Kummer's 24 solutions the

$$P^{(\alpha)}=w_1={}_2F_1(a,\,b;\,c;\,x),$$

$$P^{(\alpha')} = w_2 = x^{1-c}{}_2F_1(a-c+1,\, b-c+1;\, 2-c;\, x),$$

and

$$P^{(\gamma)} = w_3(x) = {}_2F_1(a,\, b;\, a+b-c+1;\, 1-x),$$

$$P^{(\gamma')} = w_4(x) = (1-x)^{c-a-b}{}_2F_1(c-a,\, c-b;\, c-a-b+1;\, 1-x),$$

We now let $x \to 1$ from the upper half-plane say. , Then the first equation above has

$$\begin{aligned} &P^{(\alpha)} \to \phi(a,\, b,\, c), \quad P^{(\alpha')} \to \phi((a-c+1,\, b-c+1,\, 2-c) \\ &P^{(\gamma)} \to 1, \quad P^{(\gamma')} \to 0. \end{aligned} \tag{8.12}$$

This establishes

$$\alpha_\gamma = \phi(a,\, b,\, c), \quad \alpha'_\gamma = \phi(a-c+1,\, b-c+1,\, 2-c).$$

We can then remove the restriction $\Re(c-a-b) > 0$ by analytic continuation of the parameters a, b, c.

If we assume $\Re(c-a-b) < 0$, then choosing the second of w_1 and w_2 from Kummer's list as

$$P^{(\alpha)} = (1-x)^{c-b-a}{}_2F_1(c-a,\, c-b;\, c;\, x),$$

$$P^{(\alpha')} = x^{1-c}(1-x)^{c-a-b}{}_2F_1(1-a,\, 1-b;\, 2-c;\, x)$$

and also the second of

$$P^{(\gamma)} = x^{(1-c)}{}_2F_1(a-c+1,\, b-c+1;\, a+b-c+1;\, 1-x),$$

and

$$P^{(\gamma')} = x^{1-c}(1-x)^{c-a-b}{}_2F_1(1-a,\, 1-b;\, c-a-b+1;\, 1-x).$$

Notice that as $x \to 1$, we have

$$P^{(\alpha)} \sim (1-x)^{c-b-a}\phi(c-a,\, c-b,\, c),$$

$$P^{(\alpha')} \sim (1-x)^{c-a-b}\phi(1-a,\, 1-b,\, 2-c),$$

$$P^{(\gamma)} \sim 1, \quad P^{(\gamma')} \sim (1-x)^{c-a-b}.$$

Hence we further deduce

$$\alpha_{\gamma'} = \phi(c-a,\ c-b, c), \quad \alpha'_{\gamma'} = \phi(1-a, 1-b, 2-c).$$

We then remove the restriction $\Re(c-a-b) < 0$ by analytic continuation.

Similarly, we can compute the α_β, α', etc. We record

$$\begin{pmatrix} \alpha_\gamma & \alpha_{\gamma'} \\ \alpha'_\gamma & \alpha'_{\gamma'} \end{pmatrix} = \begin{pmatrix} \phi(a,b,c) & \phi(c-a, c-b, c) \\ \phi(a-c+1, b-c+1, 2-c) & \phi(1-a, 1-b, 2-c) \end{pmatrix}$$

and

$$\begin{pmatrix} \alpha_\beta & \alpha_{\beta'} \\ \alpha'_\beta & \alpha'_{\beta'} \end{pmatrix} = \begin{pmatrix} \phi(a, c-b, c) & \phi(b, c-a, c) \\ e^{i\pi(1-c)}\phi(a-c+1, 1-b, 2-c) & e^{i\pi(1-c)}\phi(b-c+1, 1-a, 2-c) \end{pmatrix}$$

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