Dunkl operators and Clifford algebras III

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The Dunkl Dirac operator
Definitions and $\mathfrak{osp}(1|2)$ relations
Monogenics and the Fischer decomposition
A Cauchy formula

Further generalizations
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Further generalizations
Dunkl operator:

Data:
- root system $R$ in $\mathbb{R}^m$, encoding finite reflection group $G$
- multiplicity function $k : R \to \mathbb{C}$

Dunkl operators $T_i$, $i = 1, \ldots, m$

$$T_i f(x) = \partial_{x_i} f(x) + \sum_{\alpha \in R_+} k_{\alpha \alpha_i} \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle}$$

Dunkl Laplacian

$$\Delta_k = \sum_{i=1}^{m} T_i^2$$

Euler operator

$$\mathbb{E} = \sum_{i=1}^{m} x_i \partial_{x_i}$$
Definition

The Dunkl Dirac operator is defined by

\[ \mathcal{D}_k = \sum_{j=1}^{m} e_j T_j \]

with \( e_j \) the generators of \( Cl_m \).

Basic property:

\[ \mathcal{D}_k^2 = -\Delta_k \]

When \( k = 0 \), then \( \mathcal{D}_k = \partial_x \)
Consider

\[ x = \sum_{j=1}^{m} e_j x_j \]

Then one has

\[ \{ \mathcal{D}_k, x \} = -2 \left( \mathcal{E} + \frac{\mu}{2} \right), \quad \mathcal{E} = \sum_{i=1}^{m} x_i \partial x_i \]
Consider
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Then one has
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Indeed
\[ \{ D_k, x \} = \sum_{i,j} \{ e_i T_i, e_j x_j \} \]
\[ = \sum_{i,j} (e_i e_j T_i x_j + e_j e_i x_j T_i) \]
\[ = -\sum_{i} (T_i x_i + x_i T_i) + \sum_{i \neq j} e_i e_j (T_i x_j - x_j T_i). \]

and
\[ (T_i x_j - x_j T_i) f = \sum_{\alpha \in R_+} k_\alpha \alpha_i \alpha_j f(\sigma_\alpha(x)) \]
Consequence:

**Theorem**

*The operators $D_k$ and $x$ generate a Lie superalgebra, isomorphic with osp$(1|2)$, with the following relations*

\[
\begin{align*}
\{x, x\} &= -2|x|^2 \\
\{x, D_k\} &= -2 \left( E + \frac{\mu}{2} \right) \\
[|x|^2, D_k] &= -2x \\
[\Delta_k, x] &= 2D_k \\
[\Delta_k^2, |x|^2] &= 4 \left( E + \frac{\mu}{2} \right) \\
\{D_k, D_k\} &= -2\Delta_k \\
[|x|^2, D_k] &= -2\Delta_k \\
[\Delta_k, x] &= x \\
[\Delta_k, |x|^2] &= 2 |x|^2,
\end{align*}
\]

where $E = \sum_{i=1}^{m} x_i \partial_{x_i}$ is the Euler operator.
Remark:
We already know that $\Delta_k$, $|x|^2$ and $\mathbb{E} + \mu/2$ generate $\mathfrak{sl}_2$.
This follows now immediately from the previous theorem!

B. Ørsted, P. Somberg and V. Soucek,
The Howe duality for the Dunkl version of the Dirac operator.
Double cover of reflection group:

\[ \tilde{G} := p^{-1}(G) \]

Note: \( \tilde{G} \subset Pin(m) \)

Action of \( \tilde{G} \) on \( C^\infty(\mathbb{R}^m) \otimes Cl_m \) for \( s \in \tilde{G} \):

\[ \rho(s) : C^\infty(\mathbb{R}^m) \otimes Cl_m \mapsto C^\infty(\mathbb{R}^m) \otimes Cl_m \]

\[ f \otimes b \mapsto f(p(s^{-1})x) \otimes sb. \]
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\[ f \otimes b \mapsto f(\rho(s^{-1})x) \otimes sb. \]

Then:

**Proposition**

*Let \( s \in \tilde{G} \) be a reflection. Then one has the operator identities:*

\[ \rho(s)x = -x \rho(s) \]

\[ \rho(s)D_k = -D_k \rho(s). \]
\( \mathcal{P} \) is space of \( Cl_m \)-valued polynomials, i.e.

\[
\mathcal{P} = \mathbb{R}[x_1, \ldots, x_m] \otimes Cl_m
\]

Refinement of Dunkl harmonics:

**Definition**

Let \( P \) be a homogeneous polynomial of degree \( \ell \). Then we call \( P \) a Dunkl monogenic of degree \( \ell \) if

\[
\mathcal{D}_k P = 0.
\]

The space of Dunkl monogenics of degree \( \ell \):

\[ \mathcal{M}_\ell \]
Note that $\mathcal{M}_\ell \subset \mathcal{H}_\ell$

More precisely:

**Proposition**

*Let $\mu \notin -2\mathbb{N}$. One has the decomposition*

$$\mathcal{H}_\ell = \mathcal{M}_\ell \oplus x\mathcal{M}_{\ell-1}$$
Note that $\mathcal{M}_\ell \subset \mathcal{H}_\ell$

More precisely:

### Proposition

Let $\mu \not\in -2\mathbb{N}$. One has the decomposition

$$\mathcal{H}_\ell = \mathcal{M}_\ell \oplus x\mathcal{M}_{\ell-1}$$

### Proof:

Observe that

$$H_\ell = \left( H_\ell + 2(\ell - 1 + \frac{\mu}{2})xD_k H_\ell \right)_{\in \mathcal{M}_\ell} - 2(\ell - 1 + \frac{\mu}{2})xD_k H_\ell_{\in x\mathcal{M}_{\ell-1}}$$
Theorem (Full Fischer decomposition)

Let \( \mu \notin -2\mathbb{N} \). The space \( \mathcal{P}_\ell \) of homogeneous polynomials of degree \( \ell \) taking values in \( \mathbb{C}l_m \) decomposes as

\[
\mathcal{P}_\ell = \bigoplus_{i=0}^{\ell} \mathcal{M}_{\ell-i}. 
\]
Graphical interpretation:

\[
\begin{align*}
P_0 & \quad P_1 & \quad P_2 & \quad P_3 & \quad P_4 & \quad P_5 \\
M_0 \rightarrow xM_0 \rightarrow x^2M_0 \rightarrow x^3M_0 \rightarrow x^4M_0 \rightarrow x^5M_0 \\
& + \\
M_1 \rightarrow xM_1 \rightarrow x^2M_1 \rightarrow x^3M_1 \rightarrow x^4M_1 \\
& + \\
M_2 \rightarrow xM_2 \rightarrow x^2M_2 \rightarrow x^3M_2 \\
& + \\
M_3 \rightarrow xM_3 \rightarrow x^2M_3 \\
M_4 \rightarrow xM_4 \\
M_5
\end{align*}
\]
Using the Dunkl Dirac operator, the dual pair

\[(\mathcal{G}, \mathfrak{sl}_2)\]

refines to

\[(\tilde{\mathcal{G}}, \mathfrak{osp}(1|2))\]
Further properties of Dunkl monogenics

**Theorem**

Let $M_\ell$ and $M_n$ be Dunkl monogenics of different degree. Then one has

\[ \int_{S^{m-1}} M_\ell(x) M_n(x) w_k(x) d\sigma(x) = 0 \]

\[ \int_{S^{m-1}} M_\ell(x) x M_n(x) w_k(x) d\sigma(x) = 0 \]

with $w_k(x) = \prod_{\alpha \in \mathbb{R}^+} |\langle \alpha, x \rangle|^{2k_\alpha}$ the associated weight.
$\mathcal{D}_k$ is surjective on $\mathcal{P}$

Hence

$$\dim \mathcal{M}_\ell = \dim \mathcal{P}_\ell - \dim \mathcal{P}_{\ell-1}$$

$$= 2^m \left( \binom{\ell + m - 1}{m - 1} - \binom{\ell + m - 2}{m - 1} \right)$$
Dunkl Cauchy formula:
Suppose $\Sigma$ invariant under action of $G$. Then
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Dunkl Cauchy formula:
Suppose \( \Sigma \) invariant under action of \( \mathcal{G} \). Then

**Theorem**

Let \( \Sigma \) be a compact oriented differentiable \( m \)-dimensional manifold in \( \mathbb{R}^m \). Let \( f \) and \( g \) be \( C^1 \)-functions. Then

\[
\int_{\partial \Sigma} f (w_k d\sigma_{\hat{x}}) \ g = \int_{\Sigma} [(f \mathcal{D}_k) g + f (\mathcal{D}_k g)] w_k \ dx
\]

with

\[
d\sigma_{\hat{x}} = \sum_{j=1}^{m} (-1)^{j+1} e_j \hat{d}x_j, \quad \hat{d}x_j = dx_1 \ldots dx_{j-1} dx_{j+1} \ldots dx_m
\]

Proof: apply Stokes’ theorem
Outline

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Further generalizations
Some recent developments in the subject:

- deformations of $\Delta_k$
- deformations of $\mathcal{D}_k$
- Dirac cohomology in Hecke algebra
- generalized Fourier transforms in Clifford analysis
Deforming the operators in $\mathbb{R}^m$

Introduce a parameter $a > 0$ and substitute

$$|x|^2 \longrightarrow |x|^a$$

$$\Delta_k \longrightarrow |x|^{2-a} \Delta_k$$

$$E + \frac{\mu}{2} \longrightarrow E + \frac{a + \mu - 2}{2}$$
Deforming the operators in $\mathbb{R}^m$

Introduce a parameter $a > 0$ and substitute

$$
|x|^2 \quad \rightarrow \quad |x|^a \\
\Delta_k \quad \rightarrow \quad |x|^{2-a}\Delta_k \\
\mathcal{E} + \frac{\mu}{2} \quad \rightarrow \quad \mathcal{E} + \frac{a + \mu - 2}{2}
$$

The $\mathfrak{sl}_2$ relations remain valid:

$$
\left[ |x|^{2-a}\Delta_k, |x|^a \right] = 2a \left( \mathcal{E} + \frac{a + \mu - 2}{2} \right)
$$

$$
\left[ |x|^{2-a}\Delta_k, \mathcal{E} + \frac{a + \mu - 2}{2} \right] = a |x|^{2-a}\Delta_k
$$

$$
\left[ |x|^a, \mathcal{E} + \frac{a + \mu - 2}{2} \right] = -a |x|^a
$$

Ben Saïd S., Kobayashi, T. and Ørsted B.,
The associated Fourier transform

\[ F_a = e^{\frac{i\pi}{2a}(|x|^{2-a} \Delta_k - |x|^a)} \]

Main question: write \( F_a \) as

\[ F_a = \int_{\mathbb{R}^m} K(x, y) f(x) h_a(x) dx \]

Here, \( h_a(x) \) is a weight naturally associated with the deformation.
The associated Fourier transform

\[ \mathcal{F}_a = e^{i\pi |x|^{2-a}\Delta_k - |x|^a} \]

Main question: write \( \mathcal{F}_a \) as

\[ \mathcal{F}_a = \int_{\mathbb{R}^m} K(x, y) f(x) h_a(x) dx \]

Here, \( h_a(x) \) is a weight naturally associated with the deformation

Resulting kernel:

- explicitly known if \( m = 1 \) or \( a = 1, 2 \)
- other values: infinite series of Bessel functions times Gegenbauer polynomials \( \rightarrow \) Boundedness?
Our aim was to solve this using Clifford algebra techniques

Question:
Does there exist a Dirac type operator $D$ such that

$$D^2 = -|x|^{2-a} \Delta_k$$
Our aim was to solve this using Clifford algebra techniques

**Question:**

Does there exist a Dirac type operator $D$ such that

$$D^2 = -|x|^{2-a} \Delta_k$$

**Answer:** NO

But

$$D = D_k + c|x|^{-2} x E, \quad c \in \mathbb{R}$$

is interesting Clifford analog of $|x|^{2-a} \Delta_k$

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H. De Bie, B. Ørsted, P. Somberg and V. Souček,
Dunkl operators and a family of realizations of $osp(1|2)$.

H. De Bie, B. Orsted, P. Somberg and V. Soucek,
The Clifford deformation of the Hermite semigroup.
Overview of possible deformations

\[ \Delta_k - |x|^2 \]

Dunkl deformation

\[ \Delta - |x|^2 \]

Clifford deformation

\[ D^2 + (1 + c)^2 |x|^a \]

\[ |x|^{2-a} \Delta - |x|^a \]

a - deformation
Dirac cohomology in Hecke algebra

Hecke algebra $\mathbb{H}$: built from

- polynomials on a vector space $V$
- reflection group $\mathcal{G}$ (i.e. root system $R$ and parameter function $k$)
Dirac cohomology in Hecke algebra

Hecke algebra $\mathbb{H}$: built from
- polynomials on a vector space $V$
- reflection group $G$ (i.e. root system $R$ and parameter function $k$)

Construct Dirac operator in $\mathbb{H} \otimes Cl(V)$

An algebraic statement concerning this operator has interesting representation-theoretic consequences

D. Barbasch, D. Ciubotaru, P. E. Trapa,
Dirac cohomology for graded affine Hecke algebras.

J.-S. Huang, P. Pandzic,
Dirac cohomology, unitary representations and a proof of a conjecture of Vogan.
Generalized Fourier transforms

Classical FT:

$$\mathcal{F} = e^{i\pi/4} (-\Delta + |x|^2)$$

Interesting generalization in Clifford analysis

$$\mathcal{F}_\pm = e^{\pm i\pi/2} \Gamma e^{i\pi/4} (-\Delta + |x|^2)$$

with

$$\Gamma = -x\partial_x - E$$

the square root of the casimir of $\mathfrak{osp}(1|2)$
Generalized Fourier transforms

Classical FT:
\[ \mathcal{F} = e^{\frac{i\pi}{4}(-\Delta + |x|^2)} \]

Interesting generalization in Clifford analysis
\[ \mathcal{F}_\pm = e^{\pm\frac{i\pi}{2}} e^{\frac{i\pi}{4}(-\Delta + |x|^2)} \]

with
\[ \Gamma = -x\partial_x - E \]

the square root of the casimir of \( \mathfrak{osp}(1|2) \)

Results
- explicit expression for integral kernel
- generalized translation operator

H. De Bie and Y. Xu,
On the Clifford-Fourier transform.
Thank you for your attention and interest!