Dunkl operators and Clifford algebras III

Hendrik De Bie

Clifford Research Group
Department of Mathematical Analysis
Ghent University

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Outline

Dunkl operator:

Data:

- root system R in $\mathbb{R}^m$, encoding finite reflection group $\mathcal{G}$
- multiplicity function $k : R \to \mathbb{C}$

Dunkl operators T_i, $i = 1, \ldots, m$

$$T_i f(x) = \partial_{x_i} f(x) + \sum_{\alpha \in R_+} k_\alpha \alpha_i \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle}$$

Dunkl Laplacian

$$\Delta_k = \sum_{i=1}^{m} T_i^2$$

Euler operator

$$\mathbb{E} = \sum_{i=1}^{m} x_i \partial_{x_i}$$

Definition

The Dunkl Dirac operator is defined by

$$\mathcal{D}_k = \sum_{j=1}^{m} e_j T_j$$

with e_j the generators of $\mathcal{C}l_m$.

Basic property:

$$\mathcal{D}_k^2 = -\Delta_k$$

When $k = 0$, then $\mathcal{D}_k = \partial_{\underline{x}}$

Consider

$$\underline{x} = \sum_{j=1}^{m} e_j x_j$$

Then one has

$$\{\mathcal{D}_k, \underline{x}\} = -2\left(\mathbb{E} + \frac{\mu}{2}\right), \qquad \mathbb{E} = \sum_{i=1}^{m} x_i \partial_{x_i}$$

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Indeed

$$\begin{aligned}\{D_k, \underline{x}\} &= \sum_{i,j} \{e_i T_i, e_j x_j\} \\ &= \sum_{i,j} (e_i e_j T_i x_j + e_j e_i x_j T_i) \\ &= -\sum_{i} (T_i x_i + x_i T_i) + \sum_{i \neq j} e_i e_j (T_i x_j - x_j T_i).\end{aligned}$$

and

$$(T_i x_j - x_j T_i) f = \sum_{\alpha \in R_+} k_\alpha \alpha_i \alpha_j f(\sigma_\alpha(x))$$

Consequence:

Theorem

The operators $\mathcal{D}_k$ and $\underline{x}$ generate a Lie superalgebra, isomorphic with $\mathfrak{osp}(1|2)$, with the following relations

$$\{\underline{x},\underline{x}\} = -2|\underline{x}|^2 \qquad \{\mathcal{D}_k,\mathcal{D}_k\} = -2\Delta_k$$

$$\{\underline{x},\mathcal{D}_k\} = -2\left(\mathbb{E}+\tfrac{\mu}{2}\right) \qquad \left[\mathbb{E}+\tfrac{\mu}{2},\mathcal{D}_k\right] = -\mathcal{D}_k$$

$$\left[|\underline{x}|^2,\mathcal{D}_k\right] = -2\underline{x} \qquad \left[\mathbb{E}+\tfrac{\mu}{2},\underline{x}\right] = \underline{x}$$

$$[\Delta_k,\underline{x}] = 2\mathcal{D}_k \qquad \left[\mathbb{E}+\tfrac{\mu}{2},\Delta_k\right] = -2\Delta_k$$

$$\left[\Delta_k^2,|\underline{x}|^2\right] = 4\left(\mathbb{E}+\tfrac{\mu}{2}\right) \qquad \left[\mathbb{E}+\tfrac{\mu}{2},|\underline{x}|^2\right] = 2|\underline{x}|^2,$$

where $\mathbb{E} = \sum_{i=1}^m x_i\partial_{x_i}$ is the Euler operator.

Remark:
We already know that Δ_k, $|x|^2$ and $\mathbb{E} + \mu/2$ generate $\mathfrak{sl}_2$
This follows now immediately from the previous theorem!

B. Ørsted, P. Somberg and V. Soucek,
The Howe duality for the Dunkl version of the Dirac operator.
Adv. appl. Clifford alg. **19** (2009), 403–415.

Double cover of reflection group:

$$\widetilde{\mathcal{G}} := p^{-1}(\mathcal{G})$$

Note: $\widetilde{\mathcal{G}} \subset Pin(m)$
Action of $\widetilde{\mathcal{G}}$ on $C^\infty(\mathbb{R}^m) \otimes \mathcal{C}l_m$ for $s \in \widetilde{\mathcal{G}}$:

$$\begin{aligned} \rho(s) : \; & C^\infty(\mathbb{R}^m) \otimes \mathcal{C}l_m \mapsto C^\infty(\mathbb{R}^m) \otimes \mathcal{C}l_m \\ & f \otimes b \mapsto f(p(s^{-1})x) \otimes sb. \end{aligned}$$

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Then:

Proposition

Let $s \in \widetilde{\mathcal{G}}$ be a reflection. Then one has the operator identities:

$$\begin{aligned} \rho(s)\,\underline{x} &= -\,\underline{x}\,\rho(s) \\ \rho(s)\,\mathcal{D}_k &= -\,\mathcal{D}_k\,\rho(s). \end{aligned}$$

$\mathcal{P}$ is space of $\mathcal{C}l_m$-valued polynomials, i.e.

$$\mathcal{P} = \mathbb{R}[x_1, \ldots, x_m] \otimes \mathcal{C}l_m$$

Refinement of Dunkl harmonics:

Definition

Let P be a homogeneous polynomial of degree ℓ. Then we call P a Dunkl monogenic of degree ℓ if

$$\mathcal{D}_k P = 0.$$

The space of Dunkl monogenics of degree ℓ:

$$\mathcal{M}_\ell$$

Note that $\mathcal{M}_\ell \subset \mathcal{H}_\ell$
More precisely:

Proposition

Let $\mu \notin -2\mathbb{N}$. One has the decomposition

$$\mathcal{H}_\ell = \mathcal{M}_\ell \oplus \underline{x}\mathcal{M}_{\ell-1}$$

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Proof:
Observe that

$$H_\ell = \underbrace{\left(H_\ell + 2(\ell - 1 + \frac{\mu}{2})\underline{x}\mathcal{D}_k H_\ell\right)}_{\in \mathcal{M}_\ell} - \underbrace{2(\ell - 1 + \frac{\mu}{2})\underline{x}\mathcal{D}_k H_\ell}_{\in \underline{x}\mathcal{M}_{\ell-1}}$$

Theorem (Full Fischer decomposition)

Let $\mu \notin -2\mathbb{N}$. The space $\mathcal{P}_\ell$ of homogeneous polynomials of degree ℓ taking values in $\mathcal{C}l_m$ decomposes as

$$\mathcal{P}_\ell = \bigoplus_{i=0}^{\ell} \underline{x}^i \mathcal{M}_{\ell-i}.$$

Graphical interpretation:

$$
\begin{array}{ccccccccccc}
\mathcal{P}_0 & & \mathcal{P}_1 & & \mathcal{P}_2 & & \mathcal{P}_3 & & \mathcal{P}_4 & & \mathcal{P}_5 \\
 & & & & & & \| & & & & \\
\mathcal{M}_0 & \rightarrow & \underline{x}\mathcal{M}_0 & \rightarrow & \underline{x}^2\mathcal{M}_0 & \rightarrow & \underline{x}^3\mathcal{M}_0 & \rightarrow & \underline{x}^4\mathcal{M}_0 & \rightarrow & \underline{x}^5\mathcal{M}_0 \\
 & & & & & & + & & & & \\
 & & \mathcal{M}_1 & \longrightarrow & \underline{x}\mathcal{M}_1 & \longrightarrow & \underline{x}^2\mathcal{M}_1 & \rightarrow & \underline{x}^3\mathcal{M}_1 & \rightarrow & \underline{x}^4\mathcal{M}_1 \\
 & & & & & & + & & & & \\
 & & & & \mathcal{M}_2 & \longrightarrow & \underline{x}\mathcal{M}_2 & \longrightarrow & \underline{x}^2\mathcal{M}_2 & \rightarrow & \underline{x}^3\mathcal{M}_2 \\
 & & & & & & + & & & & \\
 & & & & & & \mathcal{M}_3 & \longrightarrow & \underline{x}\mathcal{M}_3 & \longrightarrow & \underline{x}^2\mathcal{M}_3 \\
 & & & & & & & & \mathcal{M}_4 & \longrightarrow & \underline{x}\mathcal{M}_4 \\
 & & & & & & & & & & \mathcal{M}_5
\end{array}
$$

Using the Dunkl Dirac operator, the dual pair

$$(\mathcal{G}, \mathfrak{sl}_2)$$

refines to

$$\left(\widetilde{\mathcal{G}}, \mathfrak{osp}(1|2)\right)$$

Further properties of Dunkl monogenics

Theorem

Let M_ℓ and M_n be Dunkl monogenics of different degree. Then one has

$$\int_{\mathbb{S}^{m-1}} M_\ell(x) M_n(x) w_k(x) d\sigma(x) = 0$$

$$\int_{\mathbb{S}^{m-1}} M_\ell(x) \underline{x} M_n(x) w_k(x) d\sigma(x) = 0$$

with $w_k(x) = \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2k_\alpha}$ the associated weight.

$\mathcal{D}_k$ is surjective on $\mathcal{P}$

Hence

$$\begin{aligned}\dim \mathcal{M}_\ell &= \dim \mathcal{P}_\ell - \dim \mathcal{P}_{\ell-1} \\ &= 2^m \left(\binom{\ell+m-1}{m-1} - \binom{\ell+m-2}{m-1} \right)\end{aligned}$$

Dunkl Cauchy formula:

Suppose Σ invariant under action of $\mathcal{G}$. Then

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Theorem

Let Σ be a compact oriented differentiable m-dimensional manifold in $\mathbb{R}^m$. Let f and g be C^1-functions. Then

$$\int_{\partial\Sigma} f \left(w_k d\sigma_{\underline{x}}\right) g = \int_{\Sigma} [(f\mathcal{D}_k)g + f(\mathcal{D}_k g)] w_k dx$$

with

$$d\sigma_{\underline{x}} = \sum_{j=1}^{m} (-1)^{j+1} e_j \widehat{dx_j}, \qquad \widehat{dx_j} = dx_1 \ldots dx_{j-1} dx_{j+1} \ldots dx_m$$

Proof: apply Stokes' theorem

Outline

Some recent developments in the subject:

- deformations of Δ_k
- deformations of $\mathcal{D}_k$
- Dirac cohomology in Hecke algebra
- generalized Fourier transforms in Clifford analysis

Deforming the operators in $\mathbb{R}^m$

Introduce a parameter $a > 0$ and substitute

$$\begin{aligned} |x|^2 &\longrightarrow |x|^a \\ \Delta_k &\longrightarrow |x|^{2-a}\Delta_k \\ \mathbb{E} + \frac{\mu}{2} &\longrightarrow \mathbb{E} + \frac{a+\mu-2}{2} \end{aligned}$$

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The $\mathfrak{sl}_2$ relations remain valid:

$$\begin{aligned} \left[|x|^{2-a}\Delta_k, |x|^a\right] &= 2a\left(\mathbb{E} + \frac{a+\mu-2}{2}\right) \\ \left[|x|^{2-a}\Delta_k, \mathbb{E} + \frac{a+\mu-2}{2}\right] &= a\,|x|^{2-a}\Delta_k \\ \left[|x|^a, \mathbb{E} + \frac{a+\mu-2}{2}\right] &= -a\,|x|^a \end{aligned}$$

Ben Saïd S., Kobayashi, T. and Ørsted B.,
Laguerre semigroup and Dunkl operators. Preprint. arXiv:0907.3749, 74 pages.

The associated Fourier transform

$$\mathcal{F}_a = e^{\frac{i\pi}{2a}(|x|^{2-a}\Delta_k - |x|^a)}$$

Main question: write $\mathcal{F}_a$ as

$$\mathcal{F}_a = \int_{\mathbb{R}^m} K(x,y) f(x) h_a(x) dx$$

Here, $h_a(x)$ is a weight naturally associated with the deformation

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Resulting kernel:

- explicitly known if $m = 1$ or $a = 1, 2$
- other values: infinite series of Bessel functions times Gegenbauer polynomials $\longrightarrow$ Boundedness?

Our aim was to solve this using Clifford algebra techniques

Question:

Does there exist a Dirac type operator $\mathbf{D}$ such that

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Does there exist a Dirac type operator $\mathbf{D}$ such that

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Answer : NO

But

$$\mathbf{D} = \mathcal{D}_k + c|x|^{-2}\underline{x}\mathbb{E}, \quad c \in \mathbb{R}$$

is interesting Clifford analog of $|x|^{2-a}\Delta_k$

H. De Bie, B. Ørsted, P. Somberg and V. Souček,
Dunkl operators and a family of realizations of $\mathfrak{osp}(1|2)$.
Preprint: arXiv:0911.4725, 25 pages.

H. De Bie, B. Orsted, P. Somberg and V. Soucek,
The Clifford deformation of the Hermite semigroup.
Preprint, 27 pages, arXiv:1101.5551.

Overview of possible deformations

$$\begin{tikzcd}
 & \Delta_k - |x|^2 \\
 & \Delta - |x|^2 \arrow[u, "\text{Dunkl deformation}"'] \arrow[dl, "\text{Clifford deformation}"'] \arrow[dr, "a\text{ - deformation}"] & \\
\mathbf{D}^2 + (1+c)^2|x|^a & & |x|^{2-a}\Delta - |x|^a
\end{tikzcd}$$

Dirac cohomology in Hecke algebra

Hecke algebra $\mathbb{H}$: built from

- polynomials on a vector space V
- reflection group $\mathcal{G}$ (i.e. root system R and parameter function k)

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Construct Dirac operator in

$$\mathbb{H} \otimes \mathcal{C}l(V)$$

An algebraic statement concerning this operator has interesting representationtheoretic consequences

D. Barbasch, D. Ciubotaru, P. E. Trapa,
Dirac cohomology for graded affine Hecke algebras.
Preprint: arXiv:1006.3822, 22 pages.

J.-S. Huang, P. Pandzic,
Dirac cohomology, unitary representations and a proof of a conjecture of Vogan.
J. Amer. Math. Soc. **15** (2002), 185-202.

Generalized Fourier transforms

Classical FT:

$$\mathcal{F} = e^{\frac{i\pi}{4}(-\Delta+|x|^2)}$$

Interesting generalization in Clifford analysis

$$\mathcal{F}_{\pm} = e^{\pm\frac{i\pi}{2}\Gamma} e^{\frac{i\pi}{4}(-\Delta+|x|^2)}$$

with

$$\Gamma = -\underline{x}\partial_{\underline{x}} - \mathbb{E}$$

the square root of the casimir of $\mathfrak{osp}(1|2)$

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Results

- explicit expression for integral kernel
- generalized translation operator

H. De Bie and Y. Xu,
On the Clifford-Fourier transform.
IMRN, arXiv:1003.0689, 30 pages.

Thank you for your attention and interest!