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On a Subclass of Strongly Gamma-Starlike Functions and Quasiconformal Extensions

O pewnej podklasie funkcji mocno gamma gwiaździstych i ich rozszerzeniu quasikonforemnym

Abstract. We consider a special subclass of strongly gamma-starlike functions of order α and show that the functions in this class are strongly-starlike of order $\beta(\alpha)$. It follows from a result of Fait, Krzyż and Zygmunt that the functions in this subclass have quasiconformal extensions.

1. Let U be the class of analytic functions $f(z) = z + \sum_2^\infty a_n z^n$ defined in the unit disk $\Delta = \{z : |z| < 1\}$. The quantities $zf'(z)/f(z)$ and $1 + zf''(z)/f'(z)$ play an important role in the geometric function theory. For example, $f(\Delta)$ is a domain starlike with respect to the origin, or a convex domain according to $\mathrm{Re}\big(zf'(z)/f(z)\big) > 0$ and $\mathrm{Re}\big(1+zf''(z)/f'(z)\big) > 0$ for all $z \in \Delta$, respectively and corresponding subclasses of U will be denoted by S^* and K, resp.

The class of functions f satisfying $\mathrm{Re}\big((1-\alpha)zf'/f + \alpha(1 + f''/f')\big) > 0$ for all $z \in \Delta$ introduced by P. T. Mocanu [8] is a generalization of both classes of starlike and convex functions. Its elements were named *alpha-convex functions* and were later shown by S. S. Miller, P. T. Mocanu and M. O. Reade [7] to be starlike for all real α.

Before we proceed any further, it is necessary to recall some elementary facts. We define the principal argument of $z = re^{i\theta}$ to satisfy $-\pi < \theta \le \pi$, and we denote $\theta = \mathrm{Arg}\, z$. The principal branch of the logarithm is defined as $\mathrm{Log}\, z = \log r + i\mathrm{Arg}\, z$. We also recall that $z^\lambda = \exp(\lambda\, \mathrm{Log}\, z)$ where $\lambda \in \mathbf{C}$. We have the following facts:

$$\begin{aligned} &\mathrm{Arg}\, z^\lambda = \lambda\, \mathrm{Arg}\, z \text{ if } 0 < \lambda \le 1\,, \\ &\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) \Leftrightarrow -\pi < \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) \le \pi\,. \end{aligned} \tag{1}$$

Z. Lewandowski, S. S. Miller and E. Złotkiewicz [5] defined another subclass of U such that

$$\mathrm{Re}\big[(zf'/f)^\gamma (1 + zf''/f')^{1-\gamma}\big] > 0\,, \text{ for all } z \in \Delta\,.$$

Here γ is real and f, f' and $\mathrm{Re}\big(1+zf''(z)/f'(z)\big)$ are $\neq 0$ in $\Delta\setminus\{0\}$. The functions f in this class are called *gamma-starlike functions* $\mathcal{L}_\gamma$ and they too have been proved to be starlike for all real γ. Clearly $\mathcal{L}_0 = S^*$ and $\mathcal{L}_1 = K$. In [5], the following subclass of U was also suggested.

Definition 1. Suppose α and γ are real such that $0 < \alpha \le 1$, $0 \le \gamma \le 1$ and $f \in \mathcal{L}_\gamma$ satisfies

$$\Big|(1-\gamma)\,\mathrm{Arg}(zf'/f) + \gamma\,\mathrm{Arg}(1+zf''/f')\Big| \le \alpha\frac{\pi}{2} \quad \text{for every } z\in\Delta .$$

Then we say that f belongs to the class of *strongly gamma-starlike functions of order* α, and we denote such class by $\mathcal{L}^*_\gamma(\alpha)$.

Note that $\mathcal{L}^*_\gamma(\alpha) \subset \mathcal{L}^*_\gamma(1) = \mathcal{L}_\gamma$, and so strongly gamma-starlike functions must be starlike. We shall show that any $f \in \mathcal{L}^*_\gamma(\alpha)$ is not only starlike but strongly-starlike of order β (depending on α) $S^*(\beta)$. This subclass of U is defined by

$$S^*(\beta) = \Big\{ f\in U : \big|\mathrm{Arg}(zf'/f)\big| \le \beta\frac{\pi}{2}\ ,\ \text{for all } z\in\Delta,\ 0<\beta\le 1\Big\}\ . \tag{2}$$

It has been studied by D. A. Brannan and W. E. Kirwan [1], M. Fait, J. G. Krzyż and J. Zygmunt [2], and J. Stankiewicz [9].

2. Let us now define the following subclass of $\mathcal{L}^*_\gamma(\alpha)$.

Definition 2. We define $\mathcal{G}^*_\gamma(\beta)$ as $\big\{f\in\mathcal{L}^*_\gamma(\alpha) : \alpha{=}\beta(1{+}\gamma){-}\gamma, \gamma/(1{+}\gamma) < \beta \le 1\big\}$. i.e. if $f \in \mathcal{G}^*_\gamma(\beta)$ then

$$|(1-\gamma)\,\mathrm{Arg}(zf'/f) + \gamma\,\mathrm{Arg}(1+zf''/f')| \le (\beta(1+\gamma)-\gamma)\frac{\pi}{2} \quad \text{for every } z\in\Delta\ , \tag{3}$$

where $\dfrac{\gamma}{1+\gamma} < \beta \le 1$.

Theorem 1. *We have $\mathcal{G}^*_\gamma(\beta) \subset S^*(\beta)$. In other words,* (3) *implies* (2).

The proof of the Theorem makes use of a well-known principle due to J.G. Clunie and I. S. Jack and similar to that in [5]. The proof of the Clunie–Jack principle can also be found in W. K. Hayman [3] and S. S. Miller and P. T. Mocanu [6].

Lemma 1. (I. S. Jack [4]) *Let $w(z) = b_m z^m + b_{m+1} z^{m+1} + \ldots$, $m \ge 1$ be an analytic function defined in Δ. Suppose $|w(z)|$ attains its maximal value on the disk $|z| \le r_0 < 1$ at z_0, i.e. $|w(z_0)| = \max\limits_{|z|\le r_0} |w(z)|$, $z = re^{i\theta}$. Then $z_0 w'(z_0)/w(z_0) = t \ge m \ge 1$.*

Proof of Theorem 1. We let $f \in \mathcal{G}^*_\gamma(\beta)$ and define $w(z)$ such that

$$zf'(z)/f(z) = \Big(\frac{1+w(z)}{1-w(z)}\Big)^{\beta}\ , \quad 0<\beta\le 1\ , \text{ for every } z\in\Delta\ . \tag{4}$$

Here $w(0) = 0$ and $w(z) \neq \pm 1$ is analytic in Δ. If $|w(z)| < 1$ for all $z \in \Delta$ then Theorem 1 follows from subordination. Suppose this is not the case. Then there exists a $z_0 = r_0 e^{i\theta_0} \in \Delta$ such that $|w(z)| < 1$ for $|z_0| < r_0$ and $|w(z)|$ attains its maximal value at z_0 which is equal 1. Then, by Lemma 1, we have, at z_0 that

$$z_0 w'(z_0)/w(z_0) = T \geq 1 \ , \text{ and } [1 + w(z_0)]/[1 - w(z_0)] = i\frac{\sin\theta_0}{1 - \cos\theta_0} = iS \ , \tag{5}$$

where S is a non–zero real number.

Let us rewrite the left hand side of (3) in the equivalent form:

$$J(\gamma, f(z)) := \big(zf'(z)/f(z)\big)^{1-\gamma}\big(1 + zf''(z)/f'(z)\big) \ , \quad 0 \leq \gamma \leq 1 \ . \tag{6}$$

Differentiate (6) and substitute (4) to obtain

$$J(\gamma, f(z)) = \Big(\frac{1 + w(z)}{1 - w(z)}\Big)^{(1-\gamma)\beta} \Big\{ \Big(\frac{1 + w(z)}{1 - w(z)}\Big)^{\beta} + \beta z\Big(\frac{w'(z)}{1 + w(z)} + \frac{w'(z)}{1 - w(z)}\Big)\Big\}^{\gamma} .$$

Applying Lemma 1 at z_0 we obtain

$$\begin{aligned} J(\gamma, f(z_0)) &= (iS)^{(1-\gamma)\beta}\Big\{(iS)^{\beta} + T\beta\Big(\frac{w'(z_0)}{1 + w(z_0)} + \frac{w'(z_0)}{1 - w(z_0)}\Big)\Big\}^{\gamma} \\ &= (iS)^{(1-\gamma)\beta}\Big\{(iS)^{\beta} + i\frac{T\beta}{2}\Big(S + \frac{1}{S}\Big)\Big\}^{\gamma} . \end{aligned} \tag{7}$$

Since S could be either a positive or negative number, it is necessary to consider both cases. We first consider S to be positive. Since $0 < \alpha \leq 1$, we clearly have

$$\text{Arg}\Big[z_0\frac{f'(z_0)}{f(z_0)}\Big)^{(1-\gamma)}\Big] = \text{Arg}(iS)^{\beta(1-\gamma)}$$

and

$$\text{Arg}\Big(1 + z_0\frac{f''(z_0)}{f'(z_0)}\Big)^{\gamma} = \text{Arg}\Big[(iS)^{\beta} + iT\beta(S + \frac{1}{S})\Big]^{\gamma}$$

less than $\pi/2$. We can apply (1). Thus taking the arguments of both sides in (7) we obtain

$$\text{Arg } J(\gamma, f(z)) = (1 - \gamma)\beta\big(\text{Arg}(i + \text{Arg } S)\big) + \gamma \text{ Arg } i^{\beta}\Big\{S^{\beta} + i^{1-\beta}\frac{T\beta}{2}(S + \frac{1}{S})\Big\} .$$

Now $\text{Arg } i^{\beta} = \beta\frac{\pi}{2}$ and

$$\begin{aligned} \text{Arg}\Big\{S^{\beta} + i^{1-\beta}\frac{T\beta}{2}(S + \frac{1}{S})\Big\} &= \tan^{-1}\Big(\frac{T\beta/2(S + 1/S)\sin[(1 - \beta)\pi/2]}{S^{\beta} + T\beta/2(S + 1/S)\cos[(1 - \beta)\pi/2]}\Big) \\ &< \tan^{-1}\Big(\frac{T\beta/2(S + 1/S)\sin[(1 - \beta)\pi/2]}{T\beta/2(S + 1/S)\cos[(1 - \beta)\pi/2]}\Big) \\ &= \tan^{-1}\big(\tan\big[(1 - \beta)\frac{\pi}{2}\big]\big) = (1 - \beta)\frac{\pi}{2} \ . \end{aligned}$$

Hence the sum of arguments of i^β and $S^\beta + i^{1-\beta}\frac{T\beta}{2}(S+\frac{1}{S})$ is less than or equal to $\pi/2$ and each argument is positive. Thus we have

$$|\mathrm{Arg}(J(\gamma, f(z_0)))| = \left|(1-\gamma)\beta\frac{\pi}{2} + \gamma\beta\frac{\pi}{2} + \gamma\mathrm{Arg}(S^\beta + i^{1-\beta}\frac{T\beta}{2}(S+\frac{1}{S}))\right| =$$

$$= \left|\beta\frac{\pi}{2} + \gamma\mathrm{Arg}\Big(S^\beta + \frac{T\beta}{2}(S+\frac{1}{S})\cos((1-\beta)\frac{\pi}{2}) + i\frac{T\beta}{2}(S+\frac{1}{S})\sin((1-\beta)\frac{\pi}{2})\Big)\right|$$

$$= \left|\beta\frac{\pi}{2} + \gamma\tan^{-1}\Big(\frac{T\beta/2(S+1/S)\sin[(1-\beta)\pi/2]}{S^\beta + T\beta/2(S+1/S)\cos[(1-\beta)\pi/2]}\Big)\right|$$

$$\geq |\beta\frac{\pi}{2}| - \gamma\left|\tan^{-1}\Big(\frac{T\beta/2(S+1/S)\sin[(1-\beta)\pi/2]}{S^\beta + T\beta/2(S+1/S)\cos[(1-\beta)\pi/2]}\Big)\right|$$

$$> \beta\frac{\pi}{2} - \gamma\tan^{-1}(\tan(1-\beta)\frac{\pi}{2}) = \beta\frac{\pi}{2} - \gamma(1-\beta)\frac{\pi}{2} = (\beta(1+\gamma)-\gamma)\frac{\pi}{2}.$$

We shall now consider S to be negative. Note that we may write $S = -|S| = e^{i\pi}|S|$ and hence $iS = e^{i\pi/2}|S|$. We have similarly

$$|\mathrm{Arg}(J(\gamma, f(z_0)))| = \left|\mathrm{Arg}\Big\{(e^{-i\pi/2}|S|)^{(1-\gamma)\beta}\big((e^{-i\pi/2}|S|)^\beta + e^{-i\pi/2}\frac{T\beta}{2}(|S|+\frac{1}{|S|})\big)^\gamma\Big\}\right|$$

$$= \left|-\frac{\pi}{2}(1-\gamma)\beta - \frac{\pi}{2}\gamma\beta + \gamma\mathrm{Arg}\Big\{|S|^\beta + \frac{T\beta}{2}(|S|+\frac{1}{|S|})e^{i(-\pi/2+\beta\pi/2)}\Big\}\right|$$

$$= \left|-\frac{\pi}{2}\beta + \gamma\mathrm{Arg}\Big\{|S|^\beta + \Big(\frac{T\beta}{2}(|S|+\frac{1}{|S|})\Big(\cos\frac{\pi}{2}(\beta-1) + i\sin\frac{\pi}{2}(\beta-1)\Big)\Big)\Big\}\right|$$

$$\geq |-\frac{\pi}{2}\beta| - \gamma\left|\tan^{-1}\Big(\frac{T\beta/2(|S|+1/|S|)\sin[(\beta-1)\pi/2]}{S^\beta + T\beta/2(|S|+1/|S|)\cos[(\beta-1)\pi/2]}\Big)\right|$$

$$> \frac{\pi}{2}\beta - \gamma\left|\tan^{-1}\Big(\frac{T\beta/2(|S|+1/|S|)\sin[(1-\beta)\pi/2]}{T\beta/2(|S|+1/|S|)\cos[(1-\beta)\pi/2]}\Big)\right|$$

$$> \beta\frac{\pi}{2} - \gamma\tan^{-1}(\tan(1-\beta)\frac{\pi}{2}) = \beta\frac{\pi}{2} - \gamma(1-\beta)\frac{\pi}{2} = (\beta(1+\gamma)-\gamma)\frac{\pi}{2}\ .$$

Hence, in both cases the above argument leads to contradictions at the same time. This completes the proof of the Theorem.

3. Let us quote the following

Lemma 2. (Fait, Krzyż and Zygmunt [2]) *If $f \in S^*(\alpha)$ for $0 < \alpha < 1$ then the mapping F defined by the formula*

$$F := \begin{cases} f(z) & |z| \leq 1 \\ |f(\zeta)|^2/\overline{f(1/\bar{z})} & |z| \geq 1 \end{cases},$$

where ζ satisfies the condition $|\zeta| = 1$, $\mathrm{Arg}(\zeta) = \mathrm{Arg}(f(1/\bar{z}))$, is a K-quasiconformal mapping of $\overline{\mathbb{C}}$ with $\frac{K-1}{K+1} = k \leq \sin(\alpha\frac{\pi}{2})$ almost everywhere.

Hence we obtain as an immediate deduction from the Theorem 1

Corollary . *The functions in the class $\mathcal{G}_\gamma^*(\beta)$ admit a K-quasiconformal extension to $\overline{\mathbb{C}}$ with $\frac{K-1}{K+1} = k \le \sin(\beta\frac{\pi}{2})$ almost everywhere.*

Note that $\mathcal{G}_0^*(\beta) = S^*(\beta)$ whereas $\mathcal{G}_1^*(\beta)$ is the class of functions satisfying

$$\left|\mathrm{Arg}(1 + zf''/f')\right| \le (2\beta - 1)\frac{\pi}{2} \text{ for all } z \in \Delta \ , \ \frac{1}{2} < \beta \le 1 \ .$$

It is called the class of functions *strongly-convex of order* $2\beta-1$. This condition implies that $f \in S^*(\beta)$. As we have seen that above implication valid only if $\frac{1}{2} < \beta \le 1$. This leaves out the range of $0 \le \beta \le \frac{1}{2}$. Hence it seems that the Corollary is not the best possible in the sense that it can include the missing range of β when $\gamma = 1$.

4. Theorem 1 showed that $f \in \mathcal{G}_\gamma^*(\beta) \Rightarrow f \in \mathcal{G}_0^*(\beta) = S^*(\beta)$. We now show that this is a special case of the following general inclusion statement.

Theorem 2. *If* $0 \le \eta \le \gamma$ *then* $\mathcal{G}_\gamma^*(\beta) \subseteq \mathcal{G}_\eta^*(\beta)$.

Proof. The case $\eta = 0$ has been dealt with in Theorem 1, so we only consider the case $0 < \eta \le \gamma \le 1$. By using subordination principle, we find that we do not need to use the same argument as in the proof of Theorem 1 again.

Let $f \in \mathcal{G}_\gamma^*(\beta)$ and let $\mathcal{P}$ denote the familiar class of functions p analytic in Δ satisfying the conditions $p(0) = 1$, $\mathrm{Re}\, p(z) > 0$ for $z \in \Delta$. Then $f \in \mathcal{G}_\gamma^*(\beta)$ if and only if there exists a $p_1(z) \in \mathcal{P}$ such that

$$(8) \qquad \left(zf'(z)/f(z)\right)^{1-\gamma}\left(1 + zf''(z)/f'(z)\right)^{\gamma} = p_1(z)^{\beta(1+\gamma)-\gamma} \ , \text{ for every } z \in \Delta \ .$$

By Theorem 1, $f \in S^*(\beta)$. Hence there also exists another $p_2(z) \in \mathcal{P}$ such that

$$(9) \qquad zf'(z)/f(z) = p_2(z)^{\beta} \ , \text{ for every } z \in \Delta \ .$$

Raise both sides of (8) and (9) to the power $\eta/\gamma \le 1$ $(\eta \ne 0)$ and $(1 - \eta/\gamma) \le 1$, respectively to obtain

$$(10) \quad \left(zf'(z)/f(z)\right)^{\eta/\gamma-\eta}\left(1 + zf''(z)/f'(z)\right)^{\eta} = p_1(z)^{\beta(\eta/\gamma+\eta)-\eta} \ , \text{ for every } z \in \Delta$$

and

$$(11) \qquad \left(zf'(z)/f(z)\right)^{(1-\eta/\gamma)} = p_2(z)^{\beta(1-\eta/\gamma)} \ , \text{ for every } z \in \Delta \ .$$

We now multiply (10) and (11) which results in

$$\left(zf'(z)/f(z)\right)^{1-\eta}\left(1 + zf''(z)/f'(z)\right)^{\eta} = p_1(z)^{\beta(\eta/\gamma+\eta)-\eta}p_2(z)^{\beta(1-\eta/\gamma)} := p_3(z) \ ,$$

for every $z \in \Delta$. Note that both the powers are less than 1 and $p_3(0) = 1$. Now

$$\begin{aligned}\left|\operatorname{Arg} p_3(z)\right| &= \left|\operatorname{Arg}(p_1(z)^{\beta(\eta/\gamma+\eta)-\eta} p_2(z)^{\beta(1-\eta/\gamma)})\right| \\ &\le \left(\beta(\frac{\eta}{\gamma}+\eta)-\eta\right)\left|\operatorname{Arg}(p_1(z))\right| + \beta(1-\frac{\eta}{\gamma})\left|\operatorname{Arg}(p_2(z))\right| \\ &< \left\{\left(\beta(\frac{\eta}{\gamma}+\eta)-\eta\right)+\beta(1-\frac{\eta}{\gamma})\right\}\frac{\pi}{2} \\ &= (\beta(\eta+1)-\eta)\frac{\pi}{2}\,.\end{aligned}$$

Since $f \in S^*(\beta)$, we have $\dfrac{\eta}{1+\eta} \le \dfrac{\gamma}{1+\gamma} < \beta \le 1$ as $\eta \le \gamma$. This is because $q(x) = \frac{x}{1+x}$ is an increasing function for all $x > 0$.

Thus

$$\left|\operatorname{Arg} p_3(x)\right| < (\beta(\eta+1)-\eta)\frac{\pi}{2} \le \frac{\pi}{2}$$

and so $p_3(z) \in \mathcal{P}$, i.e. $f \in G^*_\eta(\beta)$.

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REFERENCES

[1] Brannan, D. A., Kirwan, W. E., *On some classes of bounded univalent functions*, J. London Math. Soc. (2) 1 (1969), 431–443.

[2] Fait, M., Krzyż, J. G., Zygmunt, J., *Explicit quasiconformal extensions for some classes of univalent functions*, Comment. Math. Helv. 51 (1976), 279–285.

[3] Hayman, W. K., *A characterization of the maximum modulus of functions regular at the origin*, J. d'Analyse Math. 1 (1951), 155–179.

[4] Jack, I. S., *Functions starlike and convex of order α*, J. London Math. Soc. 3 (1971), 469–474.

[5] Lewandowski, Z., Miller, S. S., Złotkiewicz, E., *Gamma-starlike functions*, Ann. Univ. Mariae Curie-Skłodowska Sect. A 28 (1974), 53–58.

[6] Miller, S. S., Mocanu, P. T., *Second order differential inequalities in the complex plane*, J. Math. Analysis and Applications 65 (1978), 289–305.

[7] Miller, S. S., Mocanu, P. T., Reade, M. M., *All alpha-convex functions are univalent and starlike*, Proc. Amer. Math. Soc. 37 (1973), 533–554.

[8] Mocanu, P. T., *Une propriété de convexité généralisée dans la théorie de la représentation conforme*, Mathematica (Cluj) 11 (34) (1969), 127–133.

[9] Stankiewicz, J., *Quelques problémes extrémaux dans les classes des fonctions α-angulairement étoilées*, Ann. Univ. Mariae Curie-Sklodowska Sect. A 20 (1966), 59–75.

STRESZCZENIE

Autor rozważa pewną podklasę funkcji gamma-mocno gwiaździstych rzędu α i wykazuje, że funkcje tej klasy są kątowo gwiaździste rzędu $\beta(\alpha)$. Wynika stąd na mocy pewnego rezultatu Fait, Krzyża i Zygmuntowej, że funkcje tej klasy mają przedłużenie quasikonforemne.

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