Difference independence of the Riemann zeta function

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Abstract. It is proved that the Riemann zeta function does not
satisfy any nontrivial algebraic difference equation whose coefficients
are meromorphic functions $\phi$ with Nevanlinna characteristic satisfying
$T(r, \phi) = o(r)$ as $r \to \infty$.

Key Words: Riemann zeta function, difference equation, Nevan-
linna characteristic

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1. Introduction.

Let $\zeta(s)$ denote the Riemann zeta function, and $\Gamma(s)$ denote the Eu-
lar gamma function in the complex plane. These two important special
functions are related by the functional equation (see [7, 16]):

$$\zeta(1 - s) = 2^{1-s}\pi^{-s}\cos\left(\frac{1}{2}\pi s\right)\Gamma(s)\zeta(s).$$  (1.1)

A classical theorem of Hölder [6] concerning $\Gamma(s)$ states that $\Gamma(s)$ can
not satisfy any algebraic differential equation whose coefficients are rational

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functions [i.e. any equation of the form \( f(s, y, y', \ldots, y^{(n)}) = 0 \), where \( n \) is a nonnegative integer and where \( f \) is a polynomial in \( y, y', \ldots, y^{(n)} \) whose coefficients are rational functions of \( s \)]. Other proofs of this theorem were given in [3, 9, 11, 12]. Bank and Kaufman [1] generalized the theorem to coefficients being meromorphic functions \( \phi \) with Nevanlinna characteristic (see (1.5) below for its definition) satisfying \( T(r, \phi) = o(r) \) as \( r \to \infty \).

The question of the differential independence of \( \zeta(s) \) was touched upon by Hilbert in 1900 (see [5]). He conjectured that \( \zeta(s) \) and other functions of the same type do not satisfy algebraic differential equations with rational coefficients. It follows from Hilbert’s report that, based on Hölder’s theorem on the algebraic differential independence of \( \Gamma(s) \), he could prove the algebraic differential independence of \( \zeta(s) \). Hilbert’s conjectures were proved in [10, 13], see also [14, 15].

It is well known that the gamma function satisfies the following difference equation

\[
\Gamma(s + 1) = s\Gamma(s). \tag{1.2}
\]

A natural question is to ask whether \( \zeta(s) \) satisfies any algebraic difference equation, or doing \( \zeta(s) \) difference independent in short? More precisely, if \( \zeta(s + s_0), \zeta(s + s_1), \ldots, \zeta(s + s_m) \) satisfy any algebraic equation or not? Here \( m \) is a nonnegative integer, \( s_i, \ i = 0, 1, \ldots, m \) are distinct complex numbers. The special case that \( s_0, s_1, \ldots, s_m \) are real numbers was studied by Ostrowski [13]. He proved that \( \zeta(s + s_0), \zeta(s + s_1), \ldots, \zeta(s + s_m) \) can not satisfy any algebraic equation with rational coefficients. For ‘small’ \( s_i \)'s, Voronin [17] proved the following.

\textbf{Theorem A.} Let \( m \) be a nonnegative integer, \( s_i \in \mathbb{C}, \ i = 0, 1, \ldots, m, \ s_i \neq s_j, 0 \leq i < j \leq m, \) if

\[
|s_i| < \frac{1}{4}, \quad i = 0, 1, \ldots, m, \tag{1.3}
\]

and

\[
f(\zeta(s + s_0), \zeta(s + s_1), \ldots, \zeta(s + s_m)) = 0 \tag{1.4}
\]

identically in \( s \in \mathbb{C} \), where \( f(z_0, z_1, \ldots, z_m) \) is a continuous function, then \( f \) is identically zero.

Theorem A is not always true again for ‘large’ \( s_i \)'s, since the vectors \( (\zeta(s), \zeta(s + s_0)) \) are not dense in \( \mathbb{C}^2 \) for any given complex \( s_0 \) with \( \Re s_0 > 1 \), as follows from elementary properties of the zeta and gamma functions, where \( \Re s_0 \) means the real part of \( s_0 \).

In order to state our main result, we recall the standard notations of the Nevanlinna theory. Let \( \phi \) be a meromorphic function on the complex plane,
the Nevanlinna characteristic $T(r, \phi)$ of $\phi$ for $r \geq 0$ is defined by

$$T(r, \phi) = N(r, \phi) + m(r, \phi), \quad (1.5)$$

where

$$N(r, \phi) = \int_0^r \frac{n(t, \phi) - n(0, \phi)}{t} dt + n(0, \phi) \log r \quad (1.6)$$

is the pole counting function, $n(r, \phi)$ is the number of poles (counting multiplicities) of $\phi$ in the disc $\{z : |z| \leq r\}$, the proximity function $m(r, \phi)$ is given by

$$m(r, \phi) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |\phi(re^{i\theta})|d\theta, \quad (1.7)$$

where $\log^+ x = \max\{0, \log x\}$. We have

**Theorem 1.** The Riemann zeta function $\zeta(s)$ does not satisfy any algebraic difference equation whose coefficients are meromorphic functions $\phi$ with Nevanlinna characteristic satisfying $T(r, \phi) = o(r)$ as $r \to \infty$. That is, if

$$f(s, \zeta(s + s_0), \zeta(s + s_1), \ldots, \zeta(s + s_m)) = 0 \quad (1.8)$$

holds for all $s \in \mathbb{C}$, where $m$ is a nonnegative integer, $s_i \in \mathbb{C}$, $i = 0, 1, \ldots, m$, are distinct complex numbers, and $f$ is a polynomial in $\zeta(s + s_0), \zeta(s + s_1), \ldots, \zeta(s + s_m)$ whose coefficients are meromorphic functions $\phi(s)$ with Nevanlinna characteristic satisfying $T(r, \phi) = o(r)$ as $r \to \infty$, then $f$ is identically zero.

It is well known that a meromorphic function $\phi$ is rational if and only if $T(r, \phi) = O(\log r)$ as $r \to \infty$. Theorem 1 implies that $\zeta(s)$ cannot satisfy any algebraic difference equation whose coefficients are rational functions.

We recall that the Nevanlinna order of meromorphic function $\phi$ is defined by

$$\rho(\phi) = \limsup_{r \to \infty} \frac{\log T(r, \phi)}{\log r}. \quad (1.9)$$

It follows from Theorem 1 that

**Corollary 1.** The Riemann zeta function $\zeta(s)$ does not satisfy any algebraic difference equation whose coefficients are meromorphic functions $\phi$ with Nevanlinna order $\rho(\phi) < 1$.

It is easily seen that the Nevanlinna order of $\zeta(s)$ is $\rho(\zeta(s)) = 1$. By a similar calculation, we have

$$\rho\left(\frac{\zeta(s+1)}{\zeta(s)}\right) = 1. \quad (1.10)$$

It follows that Corollary 1 is best possible in the sense that the condition $\rho(\phi) < 1$ cannot be relaxed to $\rho(\phi) \leq 1$. 

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2. Proof of the constant coefficients case of Theorem 1.

In this section, we prove the constant coefficients case of Theorem 1.

**Theorem 2.** The Riemann zeta function $\zeta(s)$ does not satisfy any algebraic difference equation with constant coefficients. That is, if

$$f(\zeta(s + s_0), \zeta(s + s_1), \cdots, \zeta(s + s_m)) = 0$$  \hspace{1cm} (2.1)

for all $s \in \mathbb{C}$, where $m$ is a nonnegative integer, $s_i \in \mathbb{C}$, $i = 0, 1, \cdots, m$, are distinct complex numbers, $f$ is a polynomial in $\zeta(s + s_0), \zeta(s + s_1), \cdots, \zeta(s + s_m)$ whose coefficients are complex constants, then $f$ is identically zero.

To prove Theorem 2, we need the following lemma.

**Lemma 1.** Let $b_i, s_i \in \mathbb{C}$, $i = 0, 1, \cdots, m$, $s_i \neq s_j$, $0 \leq i < j \leq m$. Suppose that there exists an integer $p_0 > 0$ such that for all prime numbers $p > p_0$,

$$\sum_{i=0}^{m} b_i p^{-s_i} = 0.$$  \hspace{1cm} (2.2)

Then

$$b_i = 0, \quad i = 0, 1, \cdots, m.$$  \hspace{1cm} (2.3)

**Proof.** We construct an entire function of exponential type

$$g(z) = \sum_{i=0}^{m} b_i e^{-s_i z},$$  \hspace{1cm} (2.4)

It is therefore not difficult to see that $g(z)$ can not have more that $cr$ zeros in a disc of radius $r$, where $c$ is a constant, unless it vanishes identically. It follows from the hypothesis (2.2) that $g(z)$ vanishes at $z = \log p$ for prime $p > p_0$. We deduce from the Prime Number Theorem that the number of zeros of $g(z)$ in a disc of radius $r$ is at least

$$\frac{e^r}{r} + o\left(\frac{e^r}{r}\right) \geq cr$$

as $r \to +\infty$. Therefore $g(z)$ is identically zero and then (2.3) follows.

**Proof of Theorem 2.** Assume that $\zeta(s)$ satisfy

$$\sum_{i=1}^{N} P_i \zeta(s + s_0)^{k_0(i)} \zeta(s + s_1)^{k_1(i)} \cdots \zeta(s + s_m)^{k_m(i)} = 0,$$  \hspace{1cm} (2.5)

where $P_i$ ($i = 1, 2, \cdots, N$) are complex constants, not all are zero, $K(i) = (k_0(i), k_1(i), \cdots, k_m(i))$, $i = 1, 2, \cdots, N$, are multi-indices with all indices being nonnegative integers, such that

$$K(i) \neq K(j), \quad 1 \leq i < j \leq N.$$  \hspace{1cm} (2.6)
Here $K(i) = K(j)$ means $k_l(i) = k_l(j)$ for each $l, 0 \leq l \leq m$. Let
\[
\sum_{n=1}^{\infty} \frac{A_i(n)}{n^s} := \zeta(s + s_0)^{k_0(i)} \zeta(s + s_1)^{k_1(i)} \cdots \zeta(s + s_m)^{k_m(i)}, \quad i = 1, 2, \cdots, N,
\]  

where the Dirichlet series is convergent in the region
\[
1 - \min\{\Re s_0, \Re s_1, \cdots, \Re s_m\} < \Re s < \infty,
\]
then by the Uniqueness Theorem for Dirichlet series, we have
\[
\sum_{i=1}^{N} P_i A_i(n) = 0, \quad n = 1, 2, 3, \cdots. 
\]  

If there exists an $i', 1 \leq i' \leq N$ such that $K(i') = (0, 0, \cdots, 0)$, then there exists an $i, i \neq i'$ such that $P_i \neq 0$. (Otherwise we would have $P_{i'} \neq 0$ and $P_{i'} A_{i'}(1) = 0$. But $A_{i'}(1) = 1$, so we have a contradiction.) Since $A_i(1) = 1$, so
\[
A_{i'}(n) = 0, \quad n = 2, 3, 4, \cdots, 
\]
with $P_i, 1 \leq i \leq N, i \neq i'$ not all zero.

So, we may assume without loss of generality that
\[
K(i) \neq (0, 0, \cdots, 0), \quad 1 \leq i \leq N,
\]
and
\[
\sum_{i=1}^{N} P_i A_i(n) = 0, \quad n = 2, 3, 4, \cdots
\]
hold with $P_i, i = 1, 2, \cdots, N$, not all equal to zero.

Let $L$ be an arbitrary positive integer, $p_1, p_2, \cdots, p_L$ be distinct prime numbers. By (2.7), we have for $i = 1, 2, \cdots, N$,
\[
A_i(p_1 p_2 \cdots p_L) = \prod_{1 \leq l \leq L} (k_0(i)p_l^{-s_0} + k_1(i)p_l^{-s_1} + \cdots + k_m(i)p_l^{-s_m}).
\]

It follows from (2.8) and (2.13) that
\[
\sum_{i=1}^{N} P_i \prod_{1 \leq l \leq L} (k_0(i)p_l^{-s_0} + k_1(i)p_l^{-s_1} + \cdots + k_m(i)p_l^{-s_m}) = 0.
\]

Let $p_1, p_2, \cdots, p_{L-1}$ be fixed, and $p_L$ varies all over
\[
\{p \mid p \text{ is a prime number}, p > \max_{1 \leq i \leq L-1} p_l\}.
\]
We then have for all prime numbers \( p > \max_{1 \leq l \leq L-1} p_l \),

\[
\sum_{q=0}^{m} \left( \sum_{i=1}^{N} P_i \prod_{1 \leq l \leq L-1} (k_0(i)p_l^{-s_0} + k_1(i)p_l^{-s_1} + \cdots + k_m(i)p_l^{-s_m})k_{q_l}(i) \right) p^{-s_{q_1}} = 0. \tag{2.15}
\]

Thus by Lemma 1, we have for \( q_1 = 0, 1, \ldots, m \),

\[
\sum_{i=1}^{N} P_i \prod_{1 \leq l \leq L-2} (k_0(i)p_l^{-s_0} + k_1(i)p_l^{-s_1} + \cdots + k_m(i)p_l^{-s_m})k_{q_1}(i)k_{q_2}(i) = 0. \tag{2.16}
\]

Let \( p_1, p_2, \ldots, p_{L-2} \) be fixed, and \( p_{L-1} \) varies all over

\[ \{ p \mid p \text{ is a prime number}, \ p > \max_{1 \leq l \leq L-2} p_l \}. \]

We then have for \( q_1 = 0, 1, \ldots, m \), for all prime numbers \( p > \max_{1 \leq l \leq L-2} p_l \),

\[
\sum_{i=1}^{N} \sum_{q=0}^{m} P_i \prod_{1 \leq l \leq L-2} (k_0(i)p_l^{-s_0} + k_1(i)p_l^{-s_1} + \cdots + k_m(i)p_l^{-s_m})k_{q_1}(i)k_{q_2}(i)k_{q_3}(i)p^{-s_{q_4}} = 0. \tag{2.17}
\]

Thus by Lemma 1 again, we have for \( q_1, q_2 = 0, 1, \ldots, m \),

\[
\sum_{i=1}^{N} P_i k_{q_1}(i)k_{q_2}(i) \cdots k_{q_L}(i) = 0, \quad q_1, q_2, \ldots, q_L = 0, 1, \ldots, m. \tag{2.18}
\]

Continuing this procedure, we finally have for \( L = 1, 2, 3, \ldots \),

\[
\sum_{i=1}^{N} P_i k_{q_1}(i)k_{q_2}(i) \cdots k_{q_L}(i) = 0, \quad \forall x \in \mathbb{R}, \quad L = 1, 2, 3, \ldots. \tag{2.19}
\]

Then

\[
\sum_{i=1}^{N} P_i (k_0(i) + k_1(i)x + \cdots + k_m(i)x^m)^L
\]

\[
= \sum_{i=1}^{N} P_i \sum_{0 \leq q_1, q_2, \ldots, q_L \leq m} k_{q_1}(i)k_{q_2}(i) \cdots k_{q_L}(i)x^{q_1+q_2+\cdots+q_L}
\]

\[
= \sum_{0 \leq q_1, q_2, \ldots, q_L \leq m} x^{q_1+q_2+\cdots+q_L} \sum_{i=1}^{N} P_i k_{q_1}(i)k_{q_2}(i) \cdots k_{q_L}(i)
\]

\[
= 0, \quad \forall x \in \mathbb{R}, \quad L = 1, 2, 3, \ldots. \tag{2.20}
\]

Denote

\[
H_i(x) = k_0(i) + k_1(i)x + \cdots + k_m(i)x^m, \quad i = 1, 2, \ldots, N \tag{2.21}
\]

and

\[
H_0(x) = 0. \tag{2.22}
\]
Notice that \( P_i, \ i = 1, 2, \cdots, N, \) not all equal zero, by a well-known theorem in determinants due to Vandermonde, we have \( \forall x \in R, \) there exists \( 0 \leq i_1(x) < i_2(x) \leq N \) such that
\[
H_{i_1(x)}(x) = H_{i_2(x)}(x),
\]
(2.23)
here \( i_1(x), i_2(x) \) means the index may depend on \( x. \) Let
\[
E_{ij} = \{ x \ | H_i(x) = H_j(x) \}, \quad 0 \leq i < j \leq N,
\]
(2.24)
then
\[
\bigcup_{0 \leq i < j \leq N} E_{ij} = R.
\]
(2.25)
Hence there must exists \( 0 \leq \hat{i} < \hat{j} \leq N \) such that \( E_{\hat{i}\hat{j}} \) is a infinite set. That is, the polynomial
\[
H_{\hat{i}}(x) - H_{\hat{j}}(x)
\]
(2.26)
have infinitely many zeros. Then \( H_{\hat{i}}(x) - H_{\hat{j}}(x) \) must be the zero-polynomial, which is contradict with (2.7) and (2.12). The proof is complete. 

3. Proof of Theorem 1.

We need the following well-known lemma of Cartan [2] (see also [8]).

**Lemma 2.** Let \( z_1, z_2, \cdots, z_p \) be any finite collection of complex numbers, and let \( B > 0 \) be any given positive number. Then there exists a finite collection of closed disks \( D_1, D_2, \cdots, D_q \) with corresponding radii \( r_1, r_2, \cdots, r_q \) that satisfy
\[
r_1 + r_2 + \cdots + r_q = 2B,
\]
(3.1)
such that if \( z \notin D_j \) for \( j = 1, 2, \cdots, q, \) then there is a permutation of the points \( z_1, z_2, \cdots, z_p, \) say, \( \hat{z}_1, \hat{z}_2, \cdots, \hat{z}_p, \) that satisfies
\[
|z - \hat{z}_\mu| > B\frac{\mu}{p}, \quad \mu = 1, 2, \cdots, p,
\]
(3.2)
where the permutation may depend on \( z. \)

The following two lemmas may have their independent interest.

**Lemma 3.** Let \( \phi \) be a meromorphic function, and let \( \alpha > 1, \ \varepsilon > 0 \) be given real constant. Then there exists a set \( E \subset (1, \infty) \) that has upper logarithmic density
\[
\delta(E) := \limsup_{x \to +\infty} \frac{\int_{E \cap (1, x]} \frac{1}{t} dt}{\log x} < \varepsilon
\]
(3.3)
and constant $A > 0$ such that for all $z$ satisfying $|z| = r \notin [0, 1] \cup E$, we have
\begin{equation}
|\phi(z)| \leq e^{AT(\alpha r, \phi)}.
\end{equation}

**Remark.** Although an upper bound estimate similar to (3.4) for the meromorphic function $\phi(z)$ outside an exceptional set should have been recorded in the literature, the authors is not able to find such a reference.

**Proof.** Assume $\phi$ is not identically zero. Let $(a_\nu)_{\nu \in \mathbb{N}}$, resp. $(b_\mu)_{\mu \in \mathbb{N}}$, denote the sequence of all zeros, resp. all poles, of $\phi$, with due account of multiplicity. By the Poisson-Jensen formula, we have for $|z| = r < R$,
\begin{align*}
\log |\phi(z)| &= \frac{1}{2\pi} \int_0^{2\pi} \log |\phi(Re^{i\theta})| |\Re(\frac{Re^{i\theta} + z}{Re^{i\theta} - z})| d\theta \\
&\quad - \sum_{|a_\nu| < R} \log \left| \frac{R^2 - \bar{a}_\nu z}{R(z - a_\nu)} \right| + \sum_{|b_\mu| < R} \log \left| \frac{R^2 - \bar{b}_\mu z}{R(z - b_\mu)} \right| \\
&\leq \frac{R + r}{R - r} \cdot \frac{1}{2\pi} \int_0^{2\pi} \log |\phi(Re^{i\theta})| |d\theta| + \sum_{|b_\mu| < R} \log \left| \frac{R^2 - \bar{b}_\mu z}{R(z - b_\mu)} \right| \\
&\leq \frac{R + r}{R - r} (m(R, \phi) + m(\frac{1}{\phi})) + \sum_{|b_\mu| < R} \log \frac{R + r}{|z - b_\mu|},
\end{align*}
here we have used the following estimate (see [4])
\begin{equation}
\left| \frac{R^2 - \bar{a} z}{R(z - a)} \right| > 1, \text{ for } |z| < R, \ |a| < R.
\end{equation}

Let $R = \alpha^{\frac{1}{3}} r$, $\alpha > 1$, we get
\begin{align*}
\log |\phi(z)| &\leq A_1 T(\alpha^{\frac{1}{3}} r, \phi) + \sum_{|b_\mu| < \alpha^{\frac{1}{3}} r} \log \frac{(\alpha^{\frac{1}{3}} + 1)r}{|z - b_\mu|} \\
&\leq A_1 T(\alpha r, \phi) + \sum_{|b_\mu| < \alpha^{\frac{1}{3}} r} \log \frac{(\alpha^{\frac{1}{3}} + 1)r}{|z - b_\mu|},
\end{align*}
where $A_1 > 0$ is a constant.

Now we estimate $\sum_{|b_\mu| < \alpha^{\frac{1}{3}} r} \log \frac{(\alpha^{\frac{1}{3}} + 1)r}{|z - b_\mu|}$. We suppose that $h$ is a fixed nonnegative integer, and that $z$ is confined to the annulus
\begin{equation}
\alpha^{\frac{h}{3}} \leq |z| = r \leq \alpha^{\frac{h+1}{3}}.
\end{equation}

Set $p = n(\frac{h+2}{3}, \phi)$, $B = \varepsilon \alpha^{\frac{h}{3}}$, and apply Lemma 2 to the points $b_1, b_2, \cdots, b_p$, we obtain that there exists a finite collection of closed disks $D_1, D_2, \cdots, D_q$, whose radii has a total sum equal to $2B$, such that if $z \notin D_j$ for $j =
1, 2, \cdots, q$, then there is a permutation of the points \( b_1, b_2, \cdots, b_p \), say, \( \hat{b}_1, \hat{b}_2, \cdots, \hat{b}_p \), such that the inequalities

\[
|z - \hat{b}_\mu| > B \frac{\mu}{p}, \quad \mu = 1, 2, \cdots, p
\]  

(3.9)

hold. Notice that here \( q \) and \( D_1, D_2, \cdots, D_q \) depend on \( p \) and then depend on \( h \). Hence if \( z \notin D_j \) for \( j = 1, 2, \cdots, q \), we have form (3.8) and (3.9) that

\[
\sum_{|b_\mu| < \alpha \hat{r}} \log \frac{(\alpha \hat{r}^2 + 1)r}{|z - b_\mu|} \leq \sum_{\mu=1}^{p} \log \frac{A_2 p}{\mu} \leq \sum_{\mu=1}^{p} \log \frac{(A_2 p)^p}{\Gamma(p+1)}
\]

\[
\leq \log \left( \frac{(A_2 p)^p}{A_3 p^{\frac{\alpha r}{2}}} \right) = \log \left( \frac{(A_2 e)^p}{A_3 p^{\frac{\alpha r}{2}}} \right)
\]

\[
\leq \log A_4^p = p \log A_4 = \log A_4 \cdot n(\alpha \frac{\hat{r}}{r}, \phi)
\]

\[
\leq A_5 n(\alpha \hat{r}, \phi),
\]  

(11.11)

Denote by \( A_2 = \frac{\alpha \hat{r}}{\epsilon r} \) for simplicity, then by the Stirling’s formula and (3.8) we have

\[
\sum_{|b_\mu| < \alpha \hat{r}} \log \frac{(\alpha \hat{r}^2 + 1)r}{|z - b_\mu|} \leq \sum_{\mu=1}^{p} \log \frac{A_2 p}{\mu} = \log \frac{(A_2 p)^p}{\Gamma(p+1)}
\]

\[
\leq \log \left( \frac{(A_2 p)^p}{A_3 p^{\frac{\alpha r}{2}}} \right) = \log \left( \frac{(A_2 e)^p}{A_3 p^{\frac{\alpha r}{2}}} \right)
\]

\[
\leq \log A_4^p = p \log A_4 = \log A_4 \cdot n(\alpha \frac{\hat{r}}{r}, \phi)
\]

\[
\leq A_5 n(\alpha \hat{r}, \phi),
\]  

(11.11)

here \( A_3 > 0 \), \( A_4 > 1 \), \( A_5 > 0 \) are constants independent of \( r \). For \( \alpha \hat{r} > 1 \), we have

\[
N(\alpha r, \phi) \geq \int_{\alpha \hat{r}}^{\alpha r} \frac{n(t, \phi) - n(0, \phi)}{t} \, dt + n(0, \phi) \log(\alpha r)
\]

\[
\geq n(\alpha \hat{r}, \phi) \int_{\alpha \hat{r}}^{\alpha r} \frac{dt}{t} - n(0, \phi) \int_{\alpha \hat{r}}^{\alpha r} \frac{dt}{t} + n(0, \phi) \log(\alpha r)
\]

\[
\geq n(\alpha \hat{r}, \phi) \frac{\alpha \hat{r} - \alpha \hat{r}}{\alpha r} = (1 - \alpha^{-\frac{1}{2}}) n(\alpha \hat{r}, \phi),
\]  

(11.12)

then

\[
n(\alpha \hat{r}, \phi) \leq \frac{1}{1 - \alpha^{-\frac{1}{2}}} N(\alpha r, \phi).
\]  

(11.13)

(11.11) and (11.13) yields

\[
\sum_{|b_\mu| < \alpha \hat{r}} \log \frac{(\alpha \hat{r}^2 + 1)r}{|z - b_\mu|} \leq A_6 N(\alpha r, \phi) \leq A_6 T(\alpha r, \phi)
\]  

(11.14)

with constant \( A_6 > 0 \).

For each \( h \), we define (it has been mentioned that \( q \) and \( D_1, D_2, \cdots, D_q \) depend on \( h \))

\[
Y_h = \{ r : \text{there exist } z \in \bigcup_{j=1}^q D_j \text{ such that } |z| = r \},
\]  

(11.15)
\[ E_h = Y_h \cap [\alpha^h, \alpha^{h+1}]. \]  \hspace{1cm} (3.16)

Then

\[ \int_{E_h} 1 \, dx \leq \int_{Y_h} 1 \, dx \leq 4B = 4\varepsilon \alpha^\frac{h}{3}. \]  \hspace{1cm} (3.17)

Set

\[ E = \bigcup_{h=0}^\infty E_h \cap (1, \infty). \]  \hspace{1cm} (3.18)

Then by (3.7) and (3.14), we have for all \( z \) satisfying \( |z| = r \notin [0, 1] \cup E \),

\[ |\phi(z)| \leq e^{AT(\alpha, r, \phi)} \]  \hspace{1cm} (3.19)

with \( A = A_1 + A_6 \).

For any \( x > 1 \), there exist nonnegative integer \( h \) such that

\[ \alpha^\frac{h}{3} < x \leq \alpha^{\frac{h+1}{3}}. \]  \hspace{1cm} (3.20)

It follows from (3.20) and (3.17) that

\[
\int_{E \cap (1, x]} \frac{1}{t} \, dt \leq \int_{E \cap (1, \alpha^{\frac{h+1}{3}}]} \frac{1}{t} \, dt = \sum_{j=0}^{h} \int_{E_j} \frac{1}{t} \, dt \leq \sum_{j=0}^{h} \frac{1}{\alpha^3} 4\varepsilon \alpha^\frac{j}{3} = 4\varepsilon (h + 1) \leq 12\varepsilon \frac{\log x}{\log \alpha} + 4\varepsilon. \]  \hspace{1cm} (3.21)

Therefore

\[ \delta(E) = \limsup_{x \to +\infty} \int_{E \cap (1, x]} \frac{1}{t} \, dt \log x < 12\varepsilon \frac{\log x}{\log \alpha}. \]  \hspace{1cm} (3.22)

Since \( \varepsilon \) is arbitrary small, the proof is completed. \( \blacksquare \)

**Lemma 4.** Let \( N \) be a positive integer. Suppose that the Dirichlet series

\[ F_i(s) = \sum_{n=1}^\infty \frac{a_i(n)}{n^s}, \quad i = 1, 2, \cdots, N \]  \hspace{1cm} (3.23)

are convergent in the region \( \sigma_0 < \sigma < \infty \), and for each \( i = 1, 2, \cdots, N \), \( \phi_i(s) \) is meromorphic function in the complex plane with Nevanlinna characteristic satisfying

\[ T(r, \phi_i) = o(r) \quad \text{as} \quad r \to \infty. \]  \hspace{1cm} (3.24)

Suppose that

\[ \sum_{i=1}^N \phi_i(s) F_i(s) = 0 \]  \hspace{1cm} (3.25)

holds identically in \( \sigma_0 < \sigma < \infty \). Then for each positive integer \( n \),

\[ \sum_{i=1}^N a_i(n) \phi_i(s) = 0 \]  \hspace{1cm} (3.26)
holds identically in the complex plane.

**Proof.** We prove by contradiction. Let \( n_0 \) be the minimal index for which
\[
\sum_{i=1}^{N} a_i(n_0) \phi_i(s)
\]
is not identically zero in the complex plane. Then by (3.23) and (3.25), we have
\[
\frac{1}{n_0^0} \sum_{i=1}^{N} a_i(n_0) \phi_i(s) + \sum_{i=1}^{N} \phi_i(s) \left[ \sum_{n=n_0+1}^{\infty} \frac{a_i(n)}{n^s} \right] = 0
\]
identically in \( \sigma_0 < \sigma < \infty \). Denote by
\[
\psi_i(s) = \frac{\phi_i(s)}{\sum_{i=1}^{N} a_i(n_0) \phi_i(s)}, \quad i = 1, 2, \ldots, N.
\]
Then we get
\[
1 = \sum_{i=1}^{N} \psi_i(s)n_0^i \left[ \sum_{n=n_0+1}^{\infty} \frac{a_i(n)}{n^s} \right]
\]
identically in \( \sigma_0 < \sigma < \infty \). By using the well-known properties of the Nevanlinna characteristic, we have from (3.24) and (3.29) that
\[
T(r, \psi_i) = o(r) \quad \text{as} \quad r \to \infty, \quad i = 1, 2, \ldots, N.
\]
Then by using Lemma 3 with \( \epsilon \) small enough, we deduce that there exists real sequence \( \max\{0, \sigma_0\} < \sigma_k \to \infty \), such that for each \( \epsilon > 0 \), there exists \( k_\epsilon > 0 \) such that for \( k > k_\epsilon \),
\[
|\psi_i(\sigma_k)| \leq e^{\epsilon \sigma_k}, \quad i = 1, 2, \ldots, N.
\]
By the general theory of Dirichlet series, there exist \( M > 0 \) independent of \( \sigma_k \) such that
\[
|\sum_{n=n_0+1}^{\infty} \frac{a_i(n)}{n^{\sigma_k}}| \leq M(n_0 + 1)^{-\sigma_k}, \quad k > k_\epsilon, \quad i = 1, 2, \ldots, N.
\]
By (3.30) (3.32) and (3.33) we have for \( k > k_\epsilon \),
\[
1 = \left| \sum_{i=1}^{N} \psi_i(\sigma_k)n_0^i \left[ \sum_{n=n_0+1}^{\infty} \frac{a_i(n)}{n^\sigma_k} \right] \right| \leq NMe^{\epsilon \sigma_k}(\frac{n_0}{n_0 + 1})^{\sigma_k} = NMe^{(\epsilon - \log \frac{n_0 + 1}{n_0})\sigma_k}.
\]
Thus we may choose \( \epsilon \) such that \( 0 < \epsilon < \log \frac{n_0 + 1}{n_0} \), then the right hand side of (5.34) tends to zero as \( k \to \infty \) and we have a contradiction.

**Proof of Theorem 1.** Assume that \( \zeta(s) \) satisfy the following difference equation
\[
\sum_{i=1}^{N} \phi_i(s)\zeta(s + s_0)^{k_0(i)}\zeta(s + s_1)^{k_1(i)} \cdots \zeta(s + s_m)^{k_m(i)} = 0,
\]
where $\phi_i(s)$, $i = 1, 2, \cdots, N$, are meromorphic functions not all are identically zero in the complex plane, with Nevanlinna characteristic satisfying

$$T(r, \phi_i) = o(r) \quad \text{as} \quad r \to \infty,$$

(3.36)

and $K(i) = (k_0(i), k_1(i), \cdots, k_m(i))$, $i = 1, 2, \cdots, N$, are multi-indices with every index being a non-negative integer such that

$$K(i) \neq K(j), \quad 1 \leq i < j \leq N.$$

(3.37)

Here $K(i) = K(j)$ means that $k_l(i) = k_l(j)$ for each $l$, $0 \leq l \leq m$. Let

$$\sum_{n=1}^{\infty} \frac{A_i(n)}{n^s} := \zeta(s + s_0)^{k_0(i)} \zeta(s + s_1)^{k_1(i)} \cdots \zeta(s + s_m)^{k_m(i)}, \quad i = 1, 2, \cdots, N,$$

(3.38)

where the Dirichlet series is convergent in the region

$$1 - \min\{\Re s_0, \Re s_1, \cdots, \Re s_m\} < \Re s < \infty.$$

By Lemma 4, we have for each positive integer $n$,

$$\sum_{i=1}^{N} A_i(n) \phi_i(s) = 0$$

(3.39)

identically in the complex plane.

Let $S$ be a fixed point such that not all of $\phi_i(S)$, $i = 1, 2, \cdots, N$, are equal to zero, then

$$\sum_{i=1}^{N} A_i(n) \phi_i(S) = 0, \quad n = 1, 2, 3, \cdots.$$

(3.40)

We deduce from (3.40) and (3.38) that

$$\sum_{i=1}^{N} \phi_i(S) \zeta(s + s_0)^{k_0(i)} \zeta(s + s_1)^{k_1(i)} \cdots \zeta(s + s_m)^{k_m(i)} = 0$$

(3.41)

holds identically in the complex plane. That is, $\zeta(s)$ satisfies a nontrival difference equation with constant coefficients. This is a contradiction to Theorem 2. This completes the proof.

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References