Proceedings of the Edinburgh Mathematical Society (1995) 38, 13–34 ©

OSCILLATION RESULTS ON $y''+Ay=0$ IN THE COMPLEX DOMAIN WITH TRANSCENDENTAL ENTIRE COEFFICIENTS WHICH HAVE EXTREMAL DEFICIENCIES

by Y. M. CHIANG

(Received 17th June 1992; revised 7th March 1994)

Let $A(z)$ be a transcendental entire function and f_1, f_2 be linearly independent solutions of

$$y''+Ay=0.$$

We prove that if $A(z)$ has Nevanlinna deficiency $\delta(0,A)=1$, then the exponent of convergence of $E:=f_1f_2$ is infinite. The theorems that we prove here are similar to those in Bank, Laine and Langley [3].

1993 *Mathematics Subject Classification*: 34A30, 30C15, 34C10.

1. Introduction

Since 1982 there have been many efforts in order to settle the following conjecture of Bank and Laine [1]:

Let $A(z)$ be a transcendental entire function of finite order which is not an integer, and let f_1 and f_2 be two linearly independent solutions of

$$y''+A(z)y=0. \tag{1.1}$$

Then $\max\{\lambda(f_1),\lambda(f_2)\}=\infty$.

Here $\lambda(f)$ is the exponent of convergence of the zero sequence of f.

They proved the above conjecture in [1] if the order of A, denoted by $\rho(A)$, is strictly less than $\frac{1}{2}$. We note that the order of any solution f to (1.1) must be infinite. Around 1984–5, Rossi [14] and Shen [15] proved independently that the conjecture also holds when $\rho(A)=\frac{1}{2}$.

From (1.1), it is elementary that for any *non-zero* complex number q, say, we have

$$N\left(r,\frac{1}{f-q}\right)\sim T(r,f),$$

as $r \to \infty$ where $N(r,(1/f))$ is the counting function which counts how many times that $f(z)=0$ in $|z| \leqq r$ (See Section 2 for more details).

On the other hand, in a series of papers by Bank, Laine and Langley ([3–6]), in which they considered problems when $A(z)$ has some growth conditions and obtained the desired conclusion. Noticeably they proved:

Theorem A [3]. *Let $n \geqq 1$, and let $P_1, \ldots, P_n$ be non-constant polynomials whose degrees are $d_1, d_2, \ldots, d_n$ respectively, and suppose that for $i \neq j$,*

$$\deg(P_i - P_j) = \max\{d_i, d_j\}.$$

Let

$$A(z) = \sum_{j=1}^{n} B_j(z) e^{P_j(z)}$$

where, for each j, $B_j(z)$ is an entire function, not identically zero, of order $\rho(B_j) < d_j$. Then if f_1 and f_2 are two linearly independent solutions of (1.1), *we have* $\max\{\lambda(f_1), \lambda(f_2)\} = \infty$.

The same conclusion holds for the equation

$$y'' + (A(z) + P(z))y = 0$$

where $P(z)$ is a polynomial of degree n such that $n + 2 < 2\rho(A) = 2\max_j(d_j)$.

In fact, the above theorem is a consequence of:

Theorem B [3]. *Let $A(z)$ be a transcendental entire function of $\rho(A) < \infty$ with the following properties: there exists a set $H \subseteq \mathbf{R}$ of measure zero such that for each $\theta \notin H$ either*

(i) $\dfrac{|A(re^{i\theta})|}{r^N} \to \infty$, *as $r \to \infty$, for each $N > 0$, or*

(ii) $\int_0^\infty r|A(re^{i\theta})|\,dr < \infty$, *or*

(iii) *there exist positive real numbers K and b, and a non-negative real number n (all possibly depending on θ) such that*

$$n + 2 < 2\rho(A)$$

and

$$|A(re^{i\theta})| \leqq Kr^n \quad \text{for all } r \geqq b.$$

Then, if f_1 and f_2 are two linearly independent solutions of (1.1), *we have* $\max\{\lambda(f_1), \lambda(f_2)\} = \infty$.

With some stronger hypotheses, similar results were also proved for higher order equations, we refer the readers to [**4**], [**5**] and [**6**].

We note that Theorems A and B are sharp in certain senses, and we shall return to this problem later. To this end we also mention that if ρ is a positive integer or ∞, then there exists an entire function $A(z)$ of order ρ such that (1.1) possesses two linearly independent solutions each having no zeros (see [**2**, Theorem A]).

2. Definitions and notation

Let $\{a_n\}$ be a sequence of non-zero complex numbers whose moduli tending to infinity. Then the *exponent of convergence* of zeros of f is defined as the non-negative number

$$\lambda(f)=\inf\left\{q:\sum_1^{\infty}|a_n|^{-q}<\infty, q\in\mathbf{R}^+\right\}.$$

Let $T(r,f)=m(r,f)+N(r,f)$ be the *Nevanlinna characteristic* function of f where

$$m(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\overset{+}{\log}|f(re^{i\theta}|\,d\theta$$

is called the *proximity function* and

$$N(r,f)=\int_0^{\infty}\frac{n(t)}{t}\,dt$$

is the *counting function* of f. Here $n(t)$ denotes the number of poles of f in $|z|\leqq t$ and $\overset{+}{\log} a=\max\{0,\log a\}$.

The *order* of a meromorphic function f in $\mathbf{C}$ is defined as $\overline{\lim}_{r\to\infty}\log T(r,f)/\log r$ and the *deficient value* of f at $a\in\mathbf{C}$ is defined by

$$\delta(a,f)=1-\overline{\lim_{r\to\infty}}\frac{N(r,1/(f-a))}{\log r}.$$

For detailed explanation of the notations and the Nevanlinna theory, we refer the reader to [**10**].

Let $I=[1,\infty)$ and $F\subseteq I$, then

$$m(F(f))=\int_{t\in F(r)}dt \quad \text{and} \quad \mathrm{lm}(F(r))=\int_{t\in F(r)}\frac{1}{t}\,dt$$

where $F(r)=F\cap[1,r]$, are the *linear* and *logarithmic measures* respectively.

We also define

$$\mathrm{ULD}(F)=\overline{\lim_{r\to\infty}}\frac{\mathrm{lm}(F(r))}{\log r} \quad \text{and} \quad \mathrm{LLD}(F)=\underline{\lim_{r\to\infty}}\frac{\mathrm{lm}(F(r))}{\log r}$$

to be respectively the *upper* and *lower logarithmic densities.* Note that

$$\mathrm{ULD}(I)=1=\mathrm{LLD}(I) \quad \text{and} \quad \mathrm{ULD}(F)=1-\mathrm{LLD}(I\backslash F).$$

3. Main results

We consider the equation (1.1) where $A(z)$ is transcendental entire which has growth conditions similar to those of e^P where P is a polynominal as in Theorems A or B, such that the same conclusion will hold. Some results of Edrei and Fuchs suggest that for entire function $A(z)$, if $A(z)$ omits 0 'too often', for example if it has Nevanlinna deficiency $\delta(0,A)=1$ or $N(r,0)=o(T(r,A))$, then $A(z)$ behaves like e^P in certain sections of large annuli in the complex plane. In fact, they found

$$T(r,A)=O\left(\frac{|c(r)|r^P}{2}\right)$$

for r sufficiently large, where $c(r)$ may diverge. Theorem 1 below is proved precisely under these hypotheses, whereas Theorem 2 is less obvious.

Theorem 1. *Let $A(z)$ be a transcendental entire function with finite order $\rho(A)>0$, satisfying $\delta(0,A)=1$.*

(a) *Suppose that f_1 and f_2 are linearly independent solutions of*

$$y''+Ay=0. \tag{3.1}$$

Then $\max\{\lambda(f_1),\lambda(f_2)\}=\infty$.

(b) *Suppose further that $P(z)\not\equiv 0$ is a polynomial of degree n and*

$$n+2<2\rho(A). \tag{3.2}$$

Then the same conclusion holds for any two linearly independent solutions of

$$y''+(A(z)+P(z))y=0.$$

Remark. Note that in the case (b) above $n\geqq 0$, hence $\rho(A)>1$.

Let $E:=e^{z^n}$ where $n\geqq 2$ is a positive integer. Then Bank and Laine **[1]** (see also **[2]**) show that E is a product of two linearly independent solutions of

$$y''-\frac{1}{4}(e^{-2z^n}+n^2z^{2n-2}+2n(2n-1)z^{n-2})y=0 \tag{3.3}$$

where $A(z)=-\frac{1}{4}e^{-2z^n}$ and $P(z)=-\frac{1}{4}(n^2z^{2n-2}+2n(2n-1)z^{n-2})$. Clearly the degree of P equals $2n-2$ and $\rho(A)=n$. This shows that Theorems A, B and Theorem 1 (b) are sharp.

The case when $\rho(A)=1$ and $P(z)\equiv$const. in the theorem is somewhat exceptional and is also a sharp condition in this sense. This is shown by the following theorem. All these examples show that Theorem 1 is sharp.

Theorem C. (Bank, Laine and Langley [3]). *Let $K=q^2/16$ where q is an odd positive integer then*

$$y''+(e^z-K)y=0 \tag{3.4}$$

has two linearly independent solutions f_1 and f_2 such that $\lambda(f_1)\leqq 1$ and $\lambda(f_2)\leqq 1$.

Actually they later showed that $\lambda(f_1)=1=\lambda(f_2)$ where q is $\geqq 1$ [4]. The converse of Theorem C is also essentially true [3]. On the other hand Langley proved:

Theorem D. [13]. *Let $P(z)$ be a non-constant polynomial, and $\alpha\in\mathbf{C}$. Then every non-trivial solution of*

$$y''+(e^{z+\alpha}+P(z))y=0$$

satisfies $\lambda(f)=\infty$.

It is the fundamental theorem of Nevanlinna that asserts any meromorphic function $f(z)$ must satisfy $\sum_{a\in\hat{\mathbf{C}}}\delta(a,f)\leqq 2$, and $\sum_{a\in\mathbf{C}}\delta(a,f)\leqq 1$ for entire function where $\hat{\mathbf{C}}=\mathbf{C}\cup\{\infty\}$. Again in [8], Edrei and Fuchs proved that an entire function satisfies $\sum_{a\in\mathbf{C}}\delta(a,f)=1$ possesses certain regularity growth conditions. We have:

Theorem 2. *Let $A(z)$ be a transcendental entire function of finite order $\rho(A)$ and $\sum_{a\in\mathbf{C}}\delta(a,A)=1$. Suppose that f_1 and f_2 are linearly independent solutions of*

$$y''+(A(z)+P(z))y=0$$

where $P(z)$ is a polynomial with degree $n\geqq 0$ such that

$$n+2<2\rho(A). \tag{3.5}$$

Then $\max\{\lambda(f_1),\lambda(f_2)\}=\infty$.

Remark. The inequality (3.5) holds whether $P(z)$ is identically zero or not. Hence $\rho(A)>1$.

Again Theorem C shows that the Theorem is sharp since $\sum \delta(e^z - K) = \delta(K) = 1$ and $\rho(e^z - K) = 1$. We also note that Theorem 2 actually includes Theorem 1 if the order of $A(z)$, $\rho(A)$ is > 1. But we still give its proof for completeness.

The results of this paper represent some improvements of the author's Ph.D. thesis written under Professor J. M. Anderson. He would like to thank Dr. J. K. Langley for many valuable discussions and the referee for his suggestions which improved the presentation of the original manuscript. This work was supported in part by a Mayer De Rothschild scholarship from University College London and the ORS Awards.

4. Preliminaries for functions with extremal deficiencies

We first note the following well-known inequality for an entire function A,

$$\sum_{a \in \mathbf{C}} \delta(0, A) \leqq \delta(0, A'). \tag{4.1}$$

Its proof can be found in [**10**, p. 104]. We now summarise the results of Edrei and Fuchs in:

Lemma 1 [8]. *Let $A(z)$ be an entire function of finite lower order and $\delta(0, A) = 1$. Then $A(z)$ has the following properties*:

(i) *The order of $A(z)$, $\rho(A) = p$ say, is an integer and $A(z)$ can be written as*

$$A(z) = z^k e^{P(z)} \lim \prod E\left(\frac{z}{a_\nu}, p\right)$$

where $E(z, p)$ is the primary factor and a_ν are the zero of $A(z)$, and $P(z) = \alpha_0 z^p + \cdots + \alpha_p$.

(ii) *Let*

$$c(r) = \alpha_0 + \frac{1}{p} \sum_{|a_\nu| \leqq r} \frac{1}{a_\nu^p}, \tag{4.2}$$

then for any $0 < \varepsilon < 1$, we have

$$T(r, A) = (1 + \eta(\varepsilon)) \frac{|c(r)| r^p}{\pi} \quad \textit{for } r > r_0, |\eta| < \varepsilon. \tag{4.3}$$

(iii) *Let $\alpha = e^{1/(1+p)}$ and $c_j = c(\alpha^j)$ where j is an integer. Then given $\varepsilon > 0$ as in* (ii) *and $0 < \delta < (1/e)$, there exists $j_0(\varepsilon)$ such that for all $j \geqq j_0(\varepsilon)$,*

$$\left|\log|A(z)| - \operatorname{Re}(c_j z^p)\right| < 4\varepsilon |c_j| r^p, \quad z \in \Gamma_j - E_j, \tag{4.4}$$

where

$$\Gamma_j=\{z=re^{i\theta}:\alpha^j\leqq r<\alpha^{j+3/2}\}$$

and E_j is a collection of a finite number of discs whose sum of radii is at most $4e\delta\alpha^{j+3/2}$.

(iv) *There exists a path $\mathscr{L}$ extending to infinity along which we have*

$$|A(z)|>e^{(\pi/16)T(r,A)},\quad r>r_0. \tag{4.5}$$

Let $\mathscr{L}^{(k)}$ be the path which is the rotation of $\mathscr{L}$ through an angle $k\pi/p$ about the origin. Then the inequality (4.5) *remains valid for*

$$\mathscr{L}^{(2)},\mathscr{L}^{(4)},\ldots;$$

whereas on

$$\mathscr{L}^{(1)},\mathscr{L}^{(3)},\ldots, \tag{4.6}$$

we have

$$|A(z)|<e^{-(\pi/16)T(r,A)},\quad r>r_0.$$

The portion of $\mathscr{L}$ intersecting with the annuli Γ_j has a rectifiable length not exceeding a constant multiple of α^j.

Lemma 2. *The collection of the exceptional sets E_j defined in Lemma* 1(iii) *has upper logarithmic density zero.*

Proof. Recall that each E_j is a collection of a finite number of discs whose sum of radii is less than $4e\delta_j\alpha^{j+3/2}$ where $0<\delta_j<1/e$ for all j. We let $q=[\log r/\log\alpha]=[(p+1)\log r]$, where $[\alpha]$ represents the integral part of $x\in\mathbf{R}$. Also since, for each E_j, δ_j can be chosen arbitrarily small then given $\eta>0$ we may assume $\delta_j<\eta$ for all j.

Let

$$G=\bigcup_j\{r=|z|:z\in E_j\}\quad\text{and}\quad G(r)=G\cap[1,r]. \tag{4.7}$$

Consider

$$\int_{t\in G(r)}\frac{dt}{t}=O(1)+\sum_{j=1}^{q}\int_{\alpha^j}^{\alpha^{j+3/2}}\frac{dt}{t}\quad(t\in G(r))$$

$$<O(1)+\sum_{j=1}^{q}\frac{1}{\alpha^j}\int_{\alpha^j}^{\alpha^{j+3/2}}dt\quad(t\in G(r))$$

$$\leqq O(1)+\sum_{j=1}^{q}\left(\frac{1}{\alpha^j}4e\delta_j\alpha^{j+3/2}\right)$$

$$=O(1)+4e\alpha^{3/2}\sum_{j=1}^{q}\delta_j$$

$$<O(1)+4e\alpha^{3/2}q\eta.$$

Hence

$$\frac{1}{\log r}\int_{t\in G(r)}\frac{dt}{t}=O\left(\frac{1}{\log r}\right)+4e\alpha^{3/2}(p+1)\eta.$$

Since η is arbitrary we may let $r\to\infty$ to obtain ULD $(G)=0$.

Let $\theta_1<\theta_2<\cdots<\theta_n<\theta_1+2\pi$ be a finite sequence of angles, and let $\{r_{1j}\}$ and $\{r_{2j}\}$ be two unbounded strictly increasing sequences of real numbers such that $r_{1j}<r_{1j+1}\leqq r_{2j}$, and $r_{1j+1}-r_{1j}\neq o(1)$ as $j\to\infty$.

We define, for a fixed $\varepsilon>0$, the following sets:

$$P_{i,\varepsilon}=\{\theta:\theta_i-\varepsilon\leqq\theta\leqq\theta_i+\varepsilon\} \tag{4.8}$$

and

$$Q_{i,\varepsilon}=\{\theta:\theta_{i-1}+\varepsilon\leqq\theta\leqq\theta_i-\varepsilon\}\quad\text{for } i=1,\dots,n. \tag{4.9}$$

Also

$$Q_{i,\varepsilon}(r_{1j},r_{2j})=\{re^{i\theta}:\theta\in Q_{i,\varepsilon}, r\in(r_{1j},r_{2j})\} \tag{4.10}$$

and

$$Q^*_{i,\varepsilon}(r_{1j},r_{2j})=\{re^{i\theta}:re^{i\theta}\in Q_{i,\varepsilon}(r_{1j},r_{2j})\quad\text{and}\quad r\notin H\cap(r_{1j},r_{2j})\} \tag{4.11}$$

for $i=1,\dots,n$, where $H\subset I$ is a set of r with upper logarithmic density strictly less than 1. Similar definitions also hold for $P_{i,\varepsilon}(r_{1j},r_{2j})$ and $P^*_{i,\varepsilon}(r_{1j},r_{2j})$ with the same H. We shall assume, for the rest of this paper, that ε is chosen so small and $|\theta_i-\theta_j|>2\varepsilon$ for $i\neq j$ so that none of the regions $Q_{i,\varepsilon}(r_{1j},r_{2j})$, i$=1,\dots,n$ is empty.

By using the above notation, it is not difficult to deduce from the above that:

Lemma 3. *Suppose that an entire function $A(z)$ satisfies the hypotheses of Lemma 1. Then given $\varepsilon>0$, there exist sequences of $Q^*_{i,\varepsilon}(r_{1j},r_{2j})$, $i=1,2,3,\dots,2p$ and $j=1,2,3,\dots$ which are connected sets for each fixed i, where*

$$Q^*_{i,\varepsilon}(r_{1j},r_{2j}):=Q_{i,\varepsilon}(r_{1j},r_{2j})\backslash E_j$$

such that in $Q^*_{i,\varepsilon}(r_{1j},r_{2j})$ either

$$\log|A(z)| > const.\ T(r,A) \text{ for } j \geqq j_a(\varepsilon) \text{ and some } i \tag{4.12}$$

or

$$\log|A(z)| < -const.\ T(r,A) \text{ for } j \geqq j_a(\varepsilon) \text{ and for the rest of } i. \tag{4.13}$$

Proof. Let $A(z)$ satisfies the hypotheses of Lemma 1 and let its order be $p \in \mathbf{N}$. Let $\varepsilon > 0$ and $0 < \delta < 1/e$ be given, there exists an $\varepsilon_1 > 0$ such that $|\cos\theta| > 5\varepsilon_1 > 0$ for $|\theta - \theta_0| \geqq \varepsilon$ where θ_0 is any zero of $\cos\theta$. It follows from Lemma 1 that there exists a sequence $\{c_j\}$ defined by (4.2) such that (4.4) is satisfied with ε replaced by ε_1 with Γ_j and E_j also as defined there.

Let $\operatorname{Re}(c_j z^p) = |c_j| r^p \cos(p\theta + \omega_j)$ where $z = re^{i\theta}$. Also let $\theta_1 < \theta_2 < \cdots < \theta_{2p}$ be the distinct zeros of $\cos(p\theta + \omega_j)$. Clearly θ_i, $i = 1, \ldots, 2p$, depend on j. Hence $|\cos(p\theta + \omega_j)| > 5\varepsilon_1 > 0$ when $|\theta - \theta_i| \geqq \varepsilon$, $i = 1, \ldots, 2p$.

Let $r_{1j} = \alpha^j$, $r_{2j} = \alpha^{j+3/2}$ and $P_{i,\varepsilon}$, $Q_{i,\varepsilon}$, $Q^*_{i,\varepsilon}(r_{1j}, r_{2j}), \ldots$ be defined by (4.8)–(4.11) where the θ_i are precisely those zeros of $\cos(p\theta + \omega_j)$ and the exceptional set of r in $Q^*_{i,\varepsilon}(r_{1j}, r_{2j})$ arising from the existence of E_j. Then the Γ_j is divided up into $Q_{i,\varepsilon}(r_{1j}, r_{2j})$ and $P_{i,\varepsilon}(r_{1j}, r_{2j})$, $i = 1, \ldots, 2p$. From (4.4) it follows that

$$\begin{aligned} \left|\log|A(z)| - \operatorname{Re}(c_j z^p)\right| &< 4\varepsilon_1 |c_j| r^p < \tfrac{4}{5} |\cos(p\theta + \omega_j)| |c_j| r^p \\ &= \tfrac{4}{5} |\operatorname{Re}(c_j z^p)| \quad \text{for } j \geqq j_a(\varepsilon) \geqq j_0(\varepsilon) \text{ say.} \end{aligned} \tag{4.14}$$

Without loss of generality, we may assume $\cos(p\theta + \omega_j) > 0$ for $\theta \in Q_{i,\varepsilon}$ and hence $\cos(p\theta + \omega_j) < 0$ for $\theta \in Q_{i+1,\varepsilon}$ for $j \geqq j_a(\varepsilon)$. Hence it follows from (4.14) that

$$\tfrac{1}{5} \operatorname{Re}(c_j z^p) < \log|A(z)| < \tfrac{9}{5} \operatorname{Re}(c_j z^p) \quad \text{for } z \in Q^*_{i,\varepsilon}(r_{1j}, r_{2j}), \quad j \geqq j_a(\varepsilon). \tag{4.15}$$

Now (4.3) yields,

$$\log|A(z)| > \frac{1}{5} |c_j| r^p \cos(p\theta + \omega_j) > \frac{1}{5} \frac{T(r,A)}{2(1+\eta)} \quad (|\eta| < \varepsilon) \tag{4.16}$$

for $z \in Q^*_{i,\varepsilon}(r_{1j}, r_{2j})$ and $j \geqq j_a(\varepsilon)$.

It follows from (4.14) that

$$-\tfrac{9}{5} |\operatorname{Re}(c_j z^p)| < \log|A(z)| < -\tfrac{1}{5} |\operatorname{Re}(c_j z^p)| < 0 \tag{4.17}$$

for $z \in Q^*_{i+1,\varepsilon}(r_{1j}, r_{2j})$ and $j \geqq j_a(\varepsilon)$. So that

$$\log|A(z)| < -\frac{1}{5} |\cos(p\theta + \omega_j)| |c_j| r^p < -\frac{1}{5} \frac{\pi T(r,A)}{2(1+\eta)} \leqq -\frac{\pi}{20} T(r,A), \quad j \geqq j_a(\varepsilon). \tag{4.18}$$

Remark. It follows from the proof above that if (4.12) holds for all odd i (resp. even) then (4.13) holds for all even i (resp. odd).

Finally we state another result of Edrei and Fuchs.

Lemma 4 [8]. *Suppose $A(z)$ is an entire function of finite order p. If*

$$\sum_{a \in \mathbf{C}} \delta(a, A) = 1,$$

then $p \geqq 1$ is an integer. In addition, there exist $s \geqq 1$ finite asymptotic values

$$\beta_1, \beta_2, \dots, \beta_s$$

such that

$$\sum_{a \neq \infty} \delta(a, A) = \sum_{i=1}^{s} \delta(\beta_i, A)$$

and each $\delta(\beta_i, A)$ is an integral multiple of $1/p$.

In particular if $A(z)$ is a real entire function, we can choose the asymptotic paths for β_i's to be straight lines from the origin.

Remark. The asymptotic paths appearing in (4.6) of Lemma 1 are the same as those in Lemma 4 (see [**8**]).

5. Preliminary discussion and lemmas required for the proof of the theorems

The method of proof consists of investigating the growth of the function $E := f_1 f_2$ where f_1 and f_2 are two linearly independent solutions of (3.1). It is shown in [**2**] that $E(z)$ satisfies both

$$4A(z) = \left(\frac{E'}{E}\right)^2 - \frac{c^2}{E^2} - 2\frac{E''}{E} \tag{5.1}$$

where c is the Wronskian of f_1 and f_2, and the third order differential equation

$$E^{(3)} + 4A(z)E' + 2A'(z)E = 0. \tag{5.2}$$

We also note that if a function $y(z)$ satisfies the equation $y^{(k)} + A(z)y = 0$ where $k \geqq 1$ and $A(z)$ is analytic in a domain $\mathscr{A}$, say. Then integrating by parts gives the following

$$y(z) = C_0 + C_1(z - z_0) + C_2(z - z_0) + \cdots$$
$$+ C_{k-1}(z - z_0)^{k-1} - \frac{1}{(k-1)!}\int_{z_0}^{z} (z - s)^{k-1} A(s) y(s)\, ds \tag{5.3}$$

where the path of integration is taken within the domain $\mathscr{A}$.

We require the following lemmas.

Lemma 5 (Fuchs [9]). *Suppose $h(z)$ is meromorphic in $\mathbf{C}$ and of finite order ρ. Then given $\xi>0$ and $0<\delta<\frac{1}{2}$, there exists a constant $K(\rho,\xi)$ and a set of positive real numbers G of lower logarithmic density at least $1-\xi$, i.e. $\mathrm{LLD}(G)\geqq 1-\xi>0$ such that if $0\leqq\theta_2-\theta_1\leqq\delta<\frac{1}{2}$ and $r\in G$, then*

$$r\int_{\theta_1}^{\theta_2}\left|\frac{h'}{h}(re^{i\theta})\right|d\theta<K(\rho,\xi)\delta\log\frac{1}{\delta}T(r,h).$$

Lemma 6 (Valiron [16]). *Let $f(z)$ be an entire function of finite order, then*

$$\left|\frac{f'}{f}(z)\right|=O(r^k),\quad r\in[0,\infty)\backslash e.$$

Here k is some positive number and e is an R-set.

Remarks. (i) An R-set always has linear measure zero by elementary calculations.

(ii) In the course of the proofs below, it may be necessary to apply Lemma 6 repeatedly. Thus the superscripts k_1, $k_2,\dots$ and q_1, $q_2,\dots$ that appear in r^{k_1}, $r^{k_2},\dots$ and r^{q_1}, $r^{q_2},\dots$ are not necessarily the same in different occurrences.

We also require:

Lemma 7. *Let $F(re^{i\theta})$ be an entire function of finite order and $F(\theta):=F(re^{i\theta})$, with r fixed, is a solution of the partial differential equation*

$$F_\theta^{(3)}(\theta)+C_2F_\theta''(\theta)+C_1F_\theta'(\theta)+C_0F(\theta)=0 \tag{5.4}$$

where C_2 is a constant, C_1 and C_0 are complex-valued functions of $re^{i\theta}$. The subscript θ of $F_\theta'(\theta)$, $F_\theta''(\theta)$ and $F_\theta^{(3)}(\theta)$ indicates the differentiation is taken with respect to θ and r is being kept fixed. Let $\eta(r)$ be an increasing function of r and

$$|C_i(re^{i\theta})|=O(\eta(r)^2),\quad r\to\infty\quad \text{for}\quad i=0,1, \tag{5.5}$$

on a subset $\theta\in[a,b)=J\subseteq[0,4\pi)$. Suppose also that

$$|F(re^{ia})|\leqq C(r),\quad \text{for } r\notin G \text{ sufficiently large}, \tag{5.6}$$

where $C(r)$ is an increasing function of r and $m(G)<\infty$.

Then any solution $F(\theta)=F(re^{i\theta})$ of the equation (5.4) satisfies

$$\log^+|F(\theta)|=\log^+|F(re^{i\theta})|=O(\log C(r)+\eta(r)+\log(r)\quad r\to\infty, \tag{5.7}$$

$r\notin G$ and for all $\theta\in J$.

Proof. It follows from Lemma 6 that

$$F'_\theta(re^{i\theta}) = F'(z)\frac{\partial z}{\partial \theta} = ire^{i\theta}F'(z),$$

and

$$F''_\theta(re^{i\theta}) = -(r^2e^{2i\theta}F''(z) + re^{i\theta}F'(z)).$$

We have

$$|F(re^{ia})| \leqq C(r), \tag{5.8}$$

hence

$$|F'_\theta(re^{ia})| = |F(re^{ia})|\left|ire^{ia}\frac{F'}{F}(re^{ia})\right| \leqq r^{k_1}C(r), \tag{5.9}$$

and

$$|F''_\theta(re^{ia})| = |F(re^{ia})|\left|r^2e^{2ia}\frac{F''}{F}(re^{ia}) + re^{ia}\frac{F'}{F}(re^{ia})\right| \leqq r^{k_2}C(r), \tag{5.10}$$

for $r \notin G$ and for some k_1 and $k_2 > 0$. Alternatively we may prove (5.9) and (5.10) by the identity $|F'_\theta| = r|F'(z)|$ and the fact that $F'_\theta(z)$ is also an entire function.

Let $k = \max(k_1, k_2)$. It is easy to check that

$$y(\theta) = r^kC(r)\exp(\eta(r)(h+\theta)), \quad \text{for some constant } h > 4\pi,$$

satisfies the third order differential equation

$$y_\theta^{(3)}(\theta) - a_2y''_\theta(\theta) - a_1y'_\theta(\theta) - a_0y(\theta) = 0, \tag{5.11}$$

where $a_2 = |C_2|$, $a_1 = \eta^2(r) - (a_2+1)\eta(r)$ and $a_0 = \eta^2(r)$ are all positive coefficients provided r is chosen sufficiently large.

Clearly, we have for r sufficiently large and $\theta \in J$

$$|C_2(r)| = a_2,$$

$$|C_1(r)| \leqq a_1 = \eta^2(r) - (a_2+1)\eta(r) < \eta^2(r)$$

$$|C_0(r)| \leqq a_0 = \eta^2(r). \tag{5.12}$$

Here (5.12) is valid and we may multiply $\eta(r)$ by a suitable constant when necessary in the definition of $y(\theta)$. It follows from (5.8)–(5.10) that

$$|F(re^{ia})|<y(a)=r^kC(r)\exp(\eta(r)(h+a)),$$

$$|F'_\theta(re^{ia})|<y'_\theta(a)=r^kC(r)\eta(r)\exp(\eta(r)(h+a)),$$

$$|F''_\theta(re^{ia})|<y''_\theta(a)=r^kC(r)\eta^2(r)\exp(\eta(r)(h+a))$$

for $r\notin G$ sufficiently large.

We apply Herold's comparison theorem ([11], Theorem 1]) to equations (5.4) and (5.11). Thus

$$|F(re^{i\theta})|\leqq r^kC(r)\exp(\eta(r)(h+\theta)),$$

$$|F'_\theta(re^{i\theta})|\leqq r^kC(r)\eta(r)\exp(\eta(r)(h+\theta)),$$

$$|F''_\theta(re^{i\theta})|\leqq r^kC(r)\eta^2(r)\exp(\eta(r)(h+\theta))$$

for $\theta\in J$ and $r\notin G$ sufficiently large. Hence

$$\overset{+}{\log}|F(re^{i\theta})|=O(\log C(r)+\eta(r)+\log r)$$

for $\theta\in J$ and $r\notin G$. This proves the lemma.

Remark. Although the lemma has a fixed interval $[a,b)$ for θ, the conclusion still holds if we consider a sequence $[a_j,b_j)$ instead, defined on a sequence of arcs with radii $\{r_j\}$, provided the modified (5.5) and (5.6) still hold.

6. Proof of Theorem 1

We shall sketch the proof first. We assume on the contrary that $E:=f_1f_2$ with $\lambda(E)<\infty$ where f_1 and f_2 are linearly independent solutions of (3.1). It is well-known [1] that

$$T(r,E)=O\left(\bar{N}\left(r,\frac{1}{E}\right)+T(r,A)+\log r\right)$$

as $r\to\infty$ except possibly for a set r of finite linear measure. But $\bar{N}(r,(1/E))=O(r^q)$ for some $q>0$, hence $\rho(E)<\infty$. From (5.1), we also deduce $\rho(E)\geqq\rho(A)$. Then we estimate

$$T(r,E)=\frac{1}{2\pi}\int_0^{2\pi}\overset{+}{\log}\,|E(re^{i\theta})|\,d\theta$$

to yield a contradiction.

In view of the results of Rossi and Shen, we may assume that $\rho(A)>1/2$. From Lemma 3 it follows that given $\varepsilon>0$, there exist sequences of $Q^*_{i,\varepsilon}(r_{1j},r_{2j})$ and hence $P^*_{i,\varepsilon}(r_{1j},r_{2j})$ for each $i=1,2,\dots,2p$ such that $A(z)$ satisfies either (4.12) or (4.13).

We divide the proof into two parts.

Part (a). We assume $P(z)\equiv 0$. Now,

$$\int_0^{2\pi}\overset{+}{\log}\,|E(re^{i\theta})|\,d\theta=\sum_{i=1}^{2p}\int_{Q_{i,\varepsilon}}\overset{+}{\log}\,|E(re^{i\theta})|\,d\theta+\sum_{i=1}^{2p}\int_{P_{i,\varepsilon}}\overset{+}{\log}\,|E(re^{i\theta})|\,d\theta,$$

for a fixed $r\in(r_{1j},r_{2j})\backslash G$ where $j\geqq j_1(\varepsilon)$, say.

Since all the estimates for $A(z)$ in the Lemma 3 are asymptotically the same, it is sufficient to consider the following integrals only,

$$\int_{Q_{i,\varepsilon}}\overset{+}{\log}\,|E(re^{i\theta})|\,d\theta+\int_{P_{i,\varepsilon}}\overset{+}{\log}\,|E(re^{i\theta})|\,d\theta+\int_{Q_{i+1,\varepsilon}}\overset{+}{\log}\,|E(re^{i\theta})|\,d\theta=I_1+I_2+I_3, \tag{6.1}$$

where $re^{i\theta}\in Q^*_{i,\varepsilon}(r_{1j},r_{2j})$, $P^*_{i,\varepsilon}(r_{1j},r_{2j})$, $Q^*_{i+1,\varepsilon}(r_{1j},r_{2j})$ respectively when j is large and $i=1,\dots,2p-1$.

We assume that $A(z)$ satisfies estimate (4.12) in $Q^*_{i,\varepsilon}(r_{1j},r_{2j})$ and (4.13) in $Q^*_{i+1,\varepsilon}(r_{1j},r_{2j})$. Let $re^{i\theta}\in Q^*_{i,\varepsilon}(r_{1j},r_{2j})$. By Lemma 6 both

$$\left|\frac{E'}{E}(re^{i\theta})\right| \quad\text{and}\quad \left|\frac{E''}{E}(re^{i\theta})\right|$$

are of order r^{q_1} for some $q_1>0$ and outside a set of r of finite linear measure. We can incorporate the exceptional set into that of $Q^*_{i,\varepsilon}(r_{1j},r_{2j})$ for each j. But $|A(re^{i\theta})|=O(r^m)$ for each $m>0$ and $re^{i\theta}\in Q^*_{i,\varepsilon}(r_{1j},r_{2j})$ as $j\to\infty$. It follows from formula (5.1) that $|E(re^{i\theta})|<\delta$ for any given $\delta>0$ provided $j\geqq j_2(\varepsilon)$. Hence

$$\overset{+}{\log}\,|E(re^{i\theta})|=0,\quad re^{i\theta}\in Q^*_{i,\varepsilon}(r_{1j},r_{2j}), \tag{6.2}$$

and $I_1=0$ for all $j\geqq j_2(\varepsilon)$.

From Lemma 2 the exceptional set G (see (4.7)) of r arising from $\bigcup_j Q^*_{i,\varepsilon}(r_{1j},r_{2j})$ has upper logarithmic density 0. We may choose an exceptional set arising from Lemma 5 with upper logarithmic density ξ such that $\xi<1$. Also the R-set of r with finite linear measure such that (6.2) holds has upper logarithmic density zero. We may, therefore, be able to find a set of r of lower logarithmic density $1-\xi>0$ so that both (6.2) and Lemma 6 (and hence (6.3) and (6.4) below) hold simultaneously. In the rest of this proof, it may again be necessary to consider some estimates which are valid outside sets of r of finite linear measures, so that the integration (6.1) can go through. As they all have upper logarithmic densities zero, we can incorporate them into the existing exceptional set without affecting those parts of the proof. We shall not mention this fact again.

Now it follows from Lemma 5 that

$$\overset{+}{\log}|E(re^{i\theta})| \leqq \overset{+}{\log}|E(re^{i(\theta_i-\varepsilon)})| + \int_{\theta_i-\varepsilon}^{\theta} r\left|\frac{E'}{E}(re^{it})\right| dt$$

$$\leqq \overset{+}{\log}|E(re^{i(\theta_i-\varepsilon)})| + \int_{P_{i,\varepsilon}} r\left|\frac{E'}{E}(re^{it})\right| dt$$

$$\leqq \overset{+}{\log}|E(re^{i(\theta_i-\varepsilon)})| + K(\rho(E),\xi)\varepsilon\log\frac{1}{\varepsilon}T(r,E), \tag{6.3}$$

provided $|\theta_i+\varepsilon-(\theta_i-\varepsilon)|=2\varepsilon<\frac{1}{2}$ or $\varepsilon<\frac{1}{4}$ and $\theta\in P_{i,\varepsilon}$. Thus

$$I_2=O\left(\varepsilon\log\frac{1}{\varepsilon}T(r,\varepsilon)\right) \tag{6.4}$$

for $re^{i\theta}\in P^*_{i,\varepsilon}(r_{1j},r_{2j})$ and for all $j\geqq j_3(\varepsilon)$ say.

We now consider I_3. It follows from (5.2), that $E(z)$ satisfies the equation

$$E^{(3)}(z)+\phi(z)E(z)=0 \tag{6.5}$$

where

$$\phi(z)=A(z)\left(4\frac{E'}{E}(z)+2\frac{A'}{A}(z)\right).$$

Also by (5.3), any solutions of (3.1) can be expressed as

$$E(z)=b_0+b_1(z-z_0)+b_2(z-z_0)^2-\frac{1}{2!}\int_{z_0}^{z}(z-t)^2A(t)E(t)\,dt \tag{6.6}$$

where the path of integration is taken within $Q_{i+1,\varepsilon}(r_{1j},r_{2j})$. We may choose $z_0=re^{i(\theta_i+\varepsilon)}$, and hence

$$b_0=E(z_0),\quad b_1=E'(z_0)=\frac{E'}{E}(z_0)E(z_0)$$

and

$$b_2=\tfrac{1}{2}E''(z_0)=\frac{1}{2}\frac{E''}{E'}(z_0)\frac{E'}{E}(z_0)E(z_0).$$

Clearly z_0 depends on j. We apply Lemma 6 and the fact that $|z-z_0|=O(r)$ and Gronwall's Lemma (see [12]) respectively to obtain

$$|E(z)|\leqq|E(z_0)|(1+O(r^{k_1})|z-z_0|+O(r^{k_2})|z-z_0|^2)$$

$$+\frac{1}{2}\int_{\theta_i+\varepsilon}^{\theta}|re^{i\theta}-re^{it}|^2|\phi(re^{it})|\,|E(re^{it})|r\,dt$$

$$\leqq|E(z_0)|(1+O(r^{k_3}))+2\int_{\theta_i+\varepsilon}^{\theta}r^3|\phi(re^{it})|\,|E(re^{it})|\,dt$$

$$\leqq|E(z_0)|(1+O(r^{k_3}))\exp\{O(r^3|\phi(re^{it})|)\}\quad\text{for } j\geqq j_4(\varepsilon).$$

Now both

$$\frac{E'}{E}\quad\text{and}\quad\frac{A'}{A}$$

are of order r^{k_4} outside a set of r of finite linear measure, and $|A(re^{i\theta})|=o(r^{-m})$ for each $m>0$, $re^{i\theta}\in Q^*_{i+1,\varepsilon}(r_{1j},r_{2j})$ and for $j\geqq j_5(\varepsilon)$ say. Hence

$$\log^+|E(re^{i\theta})|=\log^+|E(z_0)|+O(\log r)=O\left(\varepsilon\log\frac{1}{\varepsilon}T(r,E)\right)+O(\log r)\tag{6.7}$$

from (6.3), and for $re^{i\theta}\in Q^*_{i+1,\varepsilon}(r_{1j},r_{2j})$ and $j\geqq j_5(\varepsilon)$.

Thus, we have the same estimate for

$$re^{i\theta}\in Q^*_{i,\varepsilon}(r_{1j},r_{2j})\cup P^*_{i,\varepsilon}(r_{1j},r_{2j})\cup Q^*_{i+1,\varepsilon}(r_{1j},r_{2j}),$$

for $i=1,\dots,2p-1$ provided $j\geqq j_6(\varepsilon)$. Hence

$$T(r,E)=O\left(\varepsilon\log\frac{1}{\varepsilon}T(r,E)\right)+O(\log r)$$

for $r\notin(r_{1j},r_{2j})\cap G$ and $j\geqq j_6(\varepsilon)$. Since $\varepsilon>0$ is arbitrary, we may choose it small enough to obtain

$$T(r,E)=O(\log r)$$

for $r\notin(r_{1j},r_{2j})\cap G$, $j\geqq j_7(\varepsilon)$. Hence $E(z)$ is a polynomial, and it follows from (5.1) that $A(z)$ is rational. This is a contradiction. This completes the proof of Part (a).

Part (b). We assume $P\not\equiv 0$ and $\deg P=n$, $n\geqq 0$. Given the same $\varepsilon>0$ and with the same notation and arguments, we arrive at (4.15) and (4.16) for $z\in Q^*_{i,\varepsilon}(r_{1j},r_{2j})$. Also we obtain (4.17) and (4.18) for $z\in Q^*_{i+1,\varepsilon}(r_{1j},r_{2j})$. From (4.1), we have $\delta(0,A')=1$. Thus we may assume, by Lemma 1 again, that $|\cos\theta|>5\varepsilon_1>0$ and there exists a corresponding sequence $\{\tilde{c}_j\}$ (defined by (4.2) with the zeros of $A'(z)$) such that

$$\left|\log|A'(z)|-\operatorname{Re}(\tilde{c}_jz^p)\right|<4\varepsilon_1|\tilde{c}_j|r^p,\quad z\in\Gamma_j-\tilde{E}_j,\quad \text{for all } j\geqq j_b(\varepsilon) \text{ say.}$$

Here $\{\tilde{E}_j\}$ are the exceptional sets corresponding to $A'(z)$. By Lemma 2 again, ULD $(\tilde{G})=0$ where $\tilde{G}:=\bigcup_j\{r=|z|:z\in\tilde{E}_j\}$. Let $\operatorname{Re}(\tilde{c}_jz^p)=|\tilde{c}_j|r^p\cos(p\theta+\beta_j)$ say for $z\in\Gamma_j-\tilde{E}_j$.

Suppose $\cos(p\theta+\beta_j)>0$ for $\theta\in\tilde{Q}_{k,\varepsilon}$ and hence $\cos(p\theta+\beta_j)<0$, for $\theta\in\tilde{Q}_{k+1,\varepsilon}$. Thus

$$\tfrac{1}{5}\operatorname{Re}(\tilde{c}_jz^p)<\log|A'(z)|<\tfrac{9}{5}\operatorname{Re}(\tilde{c}_jz^p),\quad z\in\tilde{Q}^*_{k,\varepsilon}(r_{1j},r_{2j})\quad \text{for } j\geqq j_b(\varepsilon). \tag{6.8}$$

We may write (5.2) in the form

$$\frac{E^{(3)}}{E}+4A\frac{E'}{E}=-2A'. \tag{6.9}$$

Since we have assumed $\lambda(E)<\infty$ and hence $\rho(E)<\infty$. It follows from Lemma 6, (4.15), (6.8) and (6.9) that for each $j\geqq j_c(\varepsilon)$ say, there exists k and i such that

$$\tilde{Q}_{k,\varepsilon}(r_{1j},r_{2j})\subseteq Q_{i,\varepsilon}(r_{1j},r_{2j}),$$

and hence for each k, $\tilde{Q}_{k,\varepsilon}(r_{1j},r_{2j})=Q_{i,\varepsilon}(r_{1j},r_{2j})$ for some i and for $j\geqq j_c(\varepsilon)$.

We may renumber k so that for $j\geqq j_c(\varepsilon)$, $k=i$ for all k and i. Now define

$$Q^{**}_{i,\varepsilon}(r_{1j},r_{2j}):=Q^*_{i,\varepsilon}(r_{1j},r_{2j})\cap\tilde{Q}^*_{i,\varepsilon}(r_{1j},r_{2j})$$

for all i and $j\geqq j_c(\varepsilon)$. We deduce

$$\log|A(z)|\leqq e^{-(\pi/20)T(r,A)}\quad\text{and}\quad\log|A'(z)|\leqq e^{-(\pi/20)T(r,A')} \tag{6.10}$$

for $z \in Q_{i,\varepsilon}^{**}(r_{1j}, r_{2j})$ for all $j \geqq j_c(\varepsilon)$. This is because the exceptional sets $\bigcup E_j$ and $\bigcup \tilde{E}_j$ both have upper logarithmic densities zero, and the previous estimates in (4.15), (4.18), (6.8) and (6.10) are still valid.

Recall that $E(z)$ satisfies equation (5.2), and let

$$V(\theta) = E(re^{i\theta}),$$

with r fixed. It is routine to check that it satisfies the equation,

$$V_\theta^{(3)}(\theta) - C_2(r)V_\theta''(\theta) - C_1(r)V_\theta'(\theta) - C_0(r)V(\theta) = 0 \tag{6.11}$$

with r being fixed, where

$$C_2(r) = 3i,$$

and

$$\begin{cases} C_1(r) = 2 + 4(re^{i\theta})^2(A+P)(re^{i\theta}), \\ C_0(r) = 2i(re^{i\theta})^3(A+P)'(re^{i\theta}) \end{cases}$$

are complex-valued coefficients and functions of r.

Suppose $A(z)$ satisfies (4.13) in $Q_{i+1,\varepsilon}^{**}(r_{1j}, r_{2j})$, and hence we may assume that it satisfies (4.12) in $Q_{i,\varepsilon}^{**}(r_{1j}, r_{2j})$. It follows from Part (a) that both I_1 and I_2 are just (6.2) and (6.4) respectively for $j \geqq j_7(\varepsilon)$.

Let $n \geqq 0$ be the degree of the polynomial $P(z)$. It follows from (4.13) that

$$|C_1(r)| = O(r^{n+2}),$$

and

$$\begin{cases} |C_0(r)| = O(r^{n+2}), & \text{if } n \geqq 1; \\ |C_0(r)| = o(1), & \text{if } n = 0; \end{cases} \tag{6.12}$$

for $re^{i\theta} \in Q_{i+1,\varepsilon}^{**}(r_{1j}, r_{2j})$ and $j \geqq j_7(\varepsilon)$.

We apply Lemma 7 to equation (6.11) and take into account the remark made after Lemma 7, with $a_j = \theta_i + \varepsilon$ and $b_j = \theta_{i+1} - \varepsilon$ (clearly, i depends on j). By (6.4),

$$\log |E(re^{ia_j})| = O\left(\varepsilon \log \frac{1}{\varepsilon} T(r, E)\right)$$

or

$$|V(a_j)| = O\left(\exp\left(\varepsilon \log \frac{1}{\varepsilon} T(r, E)\right)\right) \quad \text{for } j \geqq j_8(\varepsilon) \text{ say.}$$

As in (5.9) and (5.10), we have

$$|V'_\theta(a_j)|=|E'_\theta(re^{ia_j})|=\left|ire^{ia_j}\frac{E'}{E}(re^{ia_j})\right||E(re^{ia_j})|$$

$$=O\left(r^{q_1}\exp\left(\varepsilon\log\frac{1}{\varepsilon}T(r,E)\right)\right),\quad j\geqq j_9(\varepsilon),$$

and

$$|V''_\theta(a_j)|=|E''_\theta(re^{ia_j})|=\left|ire^{ia_j}\frac{E''}{E}(re^{ia_j})+re^{ia_j}\frac{E'}{E}(re^{ia_j})\right||E(re^{ia_j})|$$

$$=O\left(r^{q_2}\exp\left(\varepsilon\log\frac{1}{\varepsilon}T(r,E)\right)\right),\quad j\geqq j_9(\varepsilon).$$

We may choose

$$C(r)=O\left(r^{q_3}\exp\left(\varepsilon\log\frac{1}{\varepsilon}T(r,E)\right)\right),\quad j\geqq j_{10}(\varepsilon),$$

for some $q_3>0$, and (5.6) is satisfied. Now although the coefficients C_1 and C_0 in (6.12) of the equation (6.11) have bounds depending on the polynomial $P(z)$, in any case it is true that they both are bounded by a constant multiple of r^{n+2} for r (and hence j) sufficiently large and for $n\geqq 0$. So we let

$$\eta(r)^2=r^{n+2},\quad n\geqq 0.$$

Then (5.4) and (5.5) are satisfied. Hence it follows from Lemma 7 that

$$\log^+|E(re^{i\theta})|=O(\log C(r)+\eta(r)+\log r)=O\left(\varepsilon\log\frac{1}{\varepsilon}T(r,E)+\eta(r)+\log r\right)$$

for $\theta\in Q_{i+1,\varepsilon}$ and $j\geqq j_{10}(\varepsilon)$.

Again $\varepsilon>0$ is arbitrary; letting $\varepsilon\to 0$, we obtain

$$\log^+|E(re^{i\theta})|=O(r^{(n+2)/2}+\log r)\quad \text{for } re^{i\theta}\in Q_{i+1,\varepsilon}(r_{1j},r_{2j}),\ j\geqq j_{11}(\varepsilon)\text{ say.}$$

Combining (6.2) and (6.4), it follows from (6.1) that

$$T(r,E)=O(r^{(n+2)/2}+\log r),\quad n\geqq 0,$$

for $re^{i\theta} \in \cup_i Q^{**}_{i,2}(r_{1j}, r_{2j})$, $j \geqq j_{11}(\varepsilon)$. Hence $\rho(E) < \rho(A)$, and this a contradiction, since $\rho(A) > 1$, $n+2 < 2\rho(A)$, and $\rho(A) \leqq \rho(E)$. This completes the proof of Part (b) and also the proof of Theorem 1.

7. Proof of Theorem 2

We only give a sketch of the proof as it is similar to that of Theorem 1. Let $A(z)$ satisfy the hypothesis of Theorem 2. From (4.1), we have $\delta(0, A') = 1$. Given $\varepsilon > 0$, with a similar argument and notation as in the proof of Theorem 1, we have (6.8) for $z \in \tilde{Q}^*_{i,\varepsilon}(r_{1j}, r_{2j})$ for some i and for $j \geqq j_d(\varepsilon)$. Again, it is clear from equation (6.9) that $A(z)$ must also have a similar growth rate as (4.15), and $I_1 = 0$ where I_1 was defined in the proof of Theorem 1.

By Lemma 1(iv), there exist paths $\mathscr{L}^{(m)}$, $m = 1, 3, \ldots, 2p-1$, such that

$$|A'(z)| < e^{-(\pi/20)T(r, A')} \quad z \in \mathscr{L}^{(m)} \text{ and } r \text{ sufficiently large.}$$

Fix $m = 1$ say, and since ε is arbitrary, we have for $j \geqq j_e(\varepsilon)$ that $A'(z)$ satisfies

$$-\tfrac{9}{5}|\operatorname{Re}(\tilde{c}_j z^p)| < \log|A'(z)| < -\tfrac{1}{5}|\operatorname{Re}(\tilde{c}_j z^p)| < 0$$

for $z \in \tilde{Q}^*_{i,\varepsilon}(r_{1j}, r_{2j})$, and that $\mathscr{L}^{(1)} \cap \tilde{Q}^*_{i,\varepsilon}(r_{1j}, r_{2j})$ is not empty. In fact, after a suitable renumbering we may write

$$\gamma_{ij} = \mathscr{L}^{(i)} \cap \tilde{Q}_{i,\varepsilon}(r_{1j}, r_{2j}), \quad j \geqq j_e(\varepsilon),$$

and the length of it is $O(\alpha^j)$ at most. On the other hand, by Lemma 4, $A(z)$ has finite asymptotic values β_i, $i = 1, 3, \ldots, 2p-1$, not necessary distinct, such that the corresponding asymptotic paths are just $\mathscr{L}^{(i)}$ by the remark after Lemma 4. By choosing $j \geqq j_f(\varepsilon)$ and $z_{ij} \in \gamma_{ij}$ we may assume $|A(z_{ij})|$ is bounded by $2|\beta_i|$, say, where β_i is the corresponding asymptotic value. So for each $z \in \tilde{Q}^*_{i,\varepsilon}(r_{1j}, r_{2j})$, $i = 1, 3, \ldots, 2p-1$ and $j \geqq j_f(\varepsilon)$, we have

$$A(z) = \int_{z_{ij}}^{z} A'(\xi)\, d\xi + A(z_{ij}),$$

where the path of integration is taken within $\tilde{Q}_{i,\varepsilon}(r_{1j}, r_{2j})$ avoiding $\tilde{E}_j$.

But

$$\left|\int_{z_{ij}}^{z} A'(\xi)\, d\xi\right| = O(\alpha^j e^{-(\pi/20)T(\alpha^j, A')}) = o(1) \tag{7.1}$$

as $j \to \infty$ (see [**8**, p. 287]).

Thus

$$|A(z)|=O(1) \quad \text{for } z\in\tilde{Q}^*_{i,\varepsilon}(r_{1j},r_{2j}), \quad j\geqq j_f(\varepsilon), \quad i=1,3,\ldots,2p-1.$$

The above arguments can be repeated for each of $\tilde{Q}^*_{i,\varepsilon}(r_{1j},r_{2j})$, $i=1,\ldots,2p$.

Hence by equations (6.11) and (6.12) the estimations on I_3 are the same as those in part (b) of Theorem 1. Whereas I_2 is exactly the same. Hence we obtain, for $r\notin(r_{1j},r_{2j})\cap\tilde{G}$,

$$T(r,E)=O(r^{(n+2)/2}+\log r), \quad \text{for } j\to\infty \text{ and } n\geqq 0.$$

And the conclusion follows as in the case of Theorem 1.

8. Concluding remarks

We note that when $P\equiv$const., $\rho(A)=1$ and $\sum\delta(a)=1$, the method of proof of Theorem 2 just fails. However as shown by Theorems C and D that the proof cannot be extended to cover this case.

In view of Theorem D, we may ask whether the conclusions of Theorems 1 and 2 still hold when the restriction on the degree of the polynomial $P(z)$ is removed. The next obvious question is to prove a stronger conclusion that $\lambda(f)=\infty$ where f is any solution of the differential equations. Finally, the same questions can be asked for the higher order differential equations, see for example [**6**].

REFERENCES

1. S. Bank and I. Laine, On the oscillation theory of $f''+Af=0$ where A is entire, *Trans. Amer. Math. Soc.* **273** (1982), 351–363.

2. S. Bank and I. Laine, On the zeros of meromorphic solutions of second order linear differential equations, *Comment. Math. Helv.* **58** (1983), 656–677.

3. S. Bank, I. Laine and J. K. Langley, On the frequency of zeros of solutions of second order linear differential equations, *Resultate Math.* **10** (1986), 8–24.

4. S. Bank, I. Laine and J. K. Langley, Oscillation results for solutions of linear differential equations in the complex domain, *Resultate Math.* **16** (1989), 3–13.

5. S. Bank and J. K. Langley, On the oscillation of solutions of certain linear differential equations in the complex domain, *Proc. Edinburgh Math. Soc.* **30** (1987), 455–469.

6. S. Bank and J. K. Langley, Oscillation theory for higher order linear differential equations with entire coefficients, *Complex Variables* **16** (1991), 163–175.

7. Y. M. Chiang, Schwarzian derivative and second order differential equations (Ph.D. thesis, Univ. of London, 1991).

8. A. Edrei and W. H. J. Fuchs, Valeurs déficientes et valeurs asymptotiques des fonctions méromorphes, *Comment. Math. Helv.* **33** (1959), 258–295.

9. W. H. J. Fuchs Proof of a conjecture of G. Pólya concerning gap series, *Illinois J. Math.* **7** (1963), 661–667.

10. W. K. Hayman, *Meromorphic functions* (Clarendon Press, Oxford, 1964).

11. H. Herold, Ein vergleichssatz für komplexe lineare Differentialgleichungen, *Math. Z.* **126** (1972), 91–94.

12. E. Hille, *Ordinary differential equations in the complex plane* (Wiley-Interscience, New York, 1976).

13. J. K. Langley, On complex oscillation and a problem of Ozawa, *Kodai Math. J.* **9** (1986), 430–439.

14. J. Rossi, Second order differential equations with transcendental coefficients, *Proc. Amer. Math. Soc.* **97** (1986), 61–66.

15. L. C. Shen, Solution to a problem of S. Bank regarding the exponent of convergence of the solutions of a differential equation $f''+Af=0$, *Kexue Tongbao* **30** (1985), 1581–1585.

16. G. Valiron, *Lectures on the general theory of integral functions*, Chelsea, New York, 1949.

Division of Mathematics
Bolton Institute of Higher Education
Deane Road
Bolton BL3 5AB
England

Current address:
Department of Mathematics
The Hong Kong University of Science and Technology
Clear Water Bay
Kowloon
Hong Kong