3.3 Find the upper and lower limits of the sequence \( \{s_n\} \) defined by Rudin’s Ex. 4

\[
s_1 = 0; \quad s_{2m} = \frac{s_{2m-1}}{2}; \quad s_{2m+1} = \frac{1}{2} + s_{2m}.
\]

**Proof** We show that

\[
(s_{2n-1}, s_{2n}) = \left( \frac{2^{n-1} - 1}{2^{n-1}}, \frac{2^{n-1} - 1}{2^n} \right), \quad n \geq 1.
\]

In fact, by the definition,

\[
(s_1, s_2) = (0, \frac{0}{2}) = (0, 0).
\]

Suppose the formula holds for \( n = k \). Then

\[
\begin{align*}
(s_{2(k+1)-1}, s_{2(k+1)}) &= \left( \frac{1}{2} + s_{2k}, \frac{s_{2k+1}}{2} \right) = \left( \frac{1}{2} + \frac{2^{k-1} - 1}{2^k}, \frac{\frac{1}{2} + s_{2k}}{2} \right) \\
&= \left( \frac{2^{k+1} - 2}{2^{k+1}}, \frac{1}{4} + \frac{1}{2} \cdot \frac{2^{k-1} - 1}{2^k} \right) \\
&= \left( \frac{2^{(k+1)-1} - 1}{2^{(k+1)-1}}, \frac{2^{(k+1)-1} - 1}{2^{k+1}} \right).
\end{align*}
\]

The proved expression for \( \{s_n\} \) gives:

\[
\lim_{n \to \infty} s_{2n-1} = 1, \quad \lim_{n \to \infty} s_{2n-1} = \frac{1}{2}.
\]

Hence, by the definitions of upper and lower limits, we have

\[
\limsup_{n \to \infty} s_n = \lim_{n \to \infty} s_n = 1, \quad \liminf_{n \to \infty} s_n = \lim_{n \to \infty} s_n = \frac{1}{2}.
\]

3.6 Prove that the convergence of \( \sum a_n \) implies the convergence of \( \sum \sqrt{n}a_n / n \), if \( a_n \geq 0 \).

**Proof** Since \( a_n \geq 0 \) and \( \sum a_n \) converges, by Theorem 3.24, the partial sums of \( \sum a_n \) form a bounded sequence. We know that \( \sum \frac{1}{n^2} \) converges, so that the partial sums of \( \sum \frac{1}{n^2} \) also form a bounded sequence. From

\[
\frac{\sqrt{n}a_n}{n} = \sqrt{\frac{a_n}{n^2}} \leq \frac{1}{2} a_n + \frac{1}{2} \frac{1}{n^2},
\]

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we have
\[ \sum_{n=1}^{k} \frac{\sqrt{a_n}}{n} \leq \sum_{n=1}^{k} \frac{1}{2} a_n + \frac{1}{2} \sum_{n=1}^{k} \frac{1}{n^2}, \]
which implies that the partial sums of \( \sum \frac{\sqrt{a_n}}{n} \) form a bounded sequence. Hence, by Theorem 3.24, \( \sum \frac{\sqrt{a_n}}{n} \) converges.

The converse cannot be true in general. For example, take \( a_n = \frac{1}{n} \). It is known that \( \sum \frac{1}{n^{3/2}} \) converges, but \( \sum \frac{1}{n} \) diverges. \( \blacksquare \)

3.9 Suppose \( a_n > 0, s_n = a_1 + \cdots + a_n, \) and \( \sum a_n \) diverges.

(a) Prove that \( \sum \frac{a_n}{1 + a_n} \) diverges.

(b) Prove that
\[ \frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}} \]
and deduce that \( \sum \frac{a_n}{s_n} \) diverges.

(c) Prove that
\[ \frac{a_n}{s_n^2} \leq \frac{1}{s_n-1} - \frac{1}{s_n} \]
and deduce that \( \sum \frac{a_n}{s_n^2} \) converges.

(d) What can be said about \( \sum \frac{a_n}{1 + na_n} \) and \( \sum \frac{a_n}{1 + n^2a_n} \)?

**Proof** Let \( \epsilon \) be given, with \( 0 < \epsilon < 1 \).

(a) First, let us show that if \( \frac{a_n}{1 + a_n} \to 0 \), then \( a_n \to 0 \). In fact, there is \( N \) such that \( n \geq N \), we have
\[ \left| \frac{a_n}{1 + a_n} \right| < \frac{\epsilon}{1 + \epsilon}. \]
This gives that, for \( n \geq N \),
\[ |a_n| \leq \frac{\epsilon}{1 + \epsilon} \cdot (1 + |a_n|), \]
or
\[ |a_n| < \epsilon. \]

Next, we prove (a). Suppose \( \sum \frac{a_n}{1 + a_n} \) converges. By Theorem 3.23, we know \( \frac{a_n}{1 + a_n} \to 0 \), so \( a_n \to 0 \). Hence, there exists \( N_1 \) such that for \( n \geq N_1, a_n < 1 \). Then
there exists $N_2$ such that for $m, n \geq N_2$, 
\[ \frac{a_m}{1 + a_n} + \cdots + \frac{a_n}{1 + a_n} < \frac{\epsilon}{2}. \]
Thus, for $m, n \geq \max\{N_1, N_2\}$, we have
\[ \frac{\epsilon}{2} > \frac{a_m}{1 + a_n} + \cdots + \frac{a_n}{1 + a_n} > \frac{a_m}{1 + 1} + \cdots + \frac{a_n}{1 + 1} = \frac{1}{2}(a_m + \cdots + a_n) > 0, \]
i.e.,
\[ 0 < a_m + \cdots + a_n < \epsilon. \]
Hence, $\sum a_n$ converges.

(b) Suppose $\sum \frac{a_n}{s_n}$ converges. There there is $N$ such that for $m, n \geq N$,
\[ 0 < \frac{a_m}{s_m} + \cdots + \frac{a_n}{s_n} < \epsilon. \]
In particular, for $m = N + 1$ and $n = n + k$, since $\{s_n\}$ is increasing, we have
\[ \epsilon > \frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} > \frac{a_{N+1}}{s_{N+k}} + \cdots + \frac{a_{N+k}}{s_{N+k}} = 1 - \frac{s_N}{s_{N+k}}. \]
Since $\sum a_n$ diverges, $s_{N+k} \to \infty$ as $k \to \infty$. We can choose $k$ large enough such that $\frac{s_N}{s_{N+k}} < 1/2$. This gives
\[ \epsilon > 1 - \frac{s_N}{s_{N+k}} > 1 - \frac{1}{2} = \frac{1}{2}, \]
which contradicts to the fact that $\epsilon$ can be arbitrarily small. Hence, $\sum \frac{a_n}{s_n}$ diverges.

(c) Again by $a_n > 0$, we know $\{s_n\}$ is increasing. Hence, for $n \geq 2$,
\[ \frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{s_n - s_{n-1}}{s_{n-1}s_n} > \frac{s_n - s_{n-1}}{s_n^2} = \frac{a_n}{s_n^2}. \]
It follows that the partial sums of the positive series are bounded:
\[ 0 < \sum_{n=2}^{k} \frac{a_n}{s_n^2} < \sum_{n=2}^{k} \left[ \frac{1}{s_{n-1}} - \frac{1}{s_n} \right] = \frac{1}{s_1} - \frac{1}{s_k} < \frac{1}{s_1}. \]
Thus, by Theorem 3.14, we know that $\sum \frac{a_n}{s_n}$ converges.

(d) The series $\sum \frac{a_n}{1 + na_n}$ may either diverge or converge. For example, take $a_n = \frac{1}{n}$.
It is clear that $a_n > 0$, and $\sum a_n$ diverges. We have
\[ \sum \frac{a_n}{1 + na_n} = \sum \frac{1/n}{2} = \frac{1}{2} \sum \frac{1}{n} = \infty. \]
On the other hand, if we take

\[ a_n = \begin{cases} 
1, & \text{if } n = k^2, \text{ where } k \text{ is a positive integer}, \\
\frac{1}{n}, & \text{otherwise},
\end{cases} \]

We know that \( a_n > 0 \), and \( \sum a_n \) diverges since \( a_{k^2} = 1 \to 0 \). For the sequence \( \{a_n\} \), we have

\[
\sum_{n=1}^{k^2} \frac{a_n}{1 + n a_n} \leq \sum_{i=1}^{k} \frac{1}{1 + i^2} + \sum_{n=1}^{k^2} \frac{1}{1 + n \cdot \frac{1}{n^2}} \\
\leq \sum_{i=1}^{k} \frac{1}{i^2} + \sum_{n=1}^{k^2} \frac{1}{n(n+1)} \\
\leq \sum_{n=1}^{\infty} \frac{1}{n^2} + \frac{1}{1 - \frac{1}{k^2}} < 1 + \sum_{n=1}^{\infty} \frac{1}{n^2}.
\]

Since \( a_n > 0 \), the last inequality implies that the partial sums of the series \( \sum \frac{a_n}{1 + n a_n} \) is bounded above, so that it converges.

The series \( \sum \frac{a_n}{1 + n^2 a_n} \) always converge, since

\[
\sum \frac{a_n}{1 + n^2 a_n} = \sum \frac{1}{n^2 + 1/a_n} < \sum \frac{1}{n^2} < \infty.
\]

3.12 Suppose \( \{p_n\} \) is a Cauchy sequence in a metric space \( X \), and some subsequence \( \{p_{n_i}\} \) converges to a point \( p \in X \). Prove that the full sequence \( \{p_n\} \) converges to \( p \).

**Proof** Let \( \epsilon > 0 \) be given. By the condition that \( \{p_n\} \) is a Cauchy sequence, there exists \( N_1 \), such that \( m, n \geq N_1 \) implies

\[ d(p_m, p_n) < \epsilon/2. \]

By the other condition that \( \{p_{n_i}\} \) converges to a point \( p \in X \), there exists \( N_2 \) such that \( i \geq N_2 \) implies

\[ d(p_{n_i}, p) < \epsilon/2. \]

We fix \( i_0 \geq \max\{N_1, N_2\} \). Thus, for any \( n \geq N_1 \), we have

\[ d(p_n, p) \leq d(p_n, p_{n_{i_0}}) + d(p_{n_{i_0}}, p) < \epsilon/2 + \epsilon/2 = \epsilon, \]

since \( n_{i_0} \geq i_0 \geq N_1 \), and \( i_0 \geq N_2 \). By the definition of limit, we know that \( \{p_n\} \) converges to \( p \). 

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