Chapter 5.

Captial Asset Pricing Model

This chapter of introduces the capital asset pricing model (Sharpe, 1964). Despite conflicting evidence reported in the Fama and French (2004), among others, its importance is marked by the simplicity of the model and its wide appliations in financial industry. We also discuss the Fama-French factor models as an extension of the CAPM.

5.1. The Capital market line, security market line and security characteristic lines

Consider a market p risky assets and a risk free asset with return μ_f. How do we invest "optimally" in the market? In the last chapter, we learned that the efficient portfolios all lie on the tangent line, which are all combination of the risk free asset and the tangent portfolio, here called market portfolio. Let R_M be the market portfolio with mean return μ_M and standard deviation σ_M, which were denoted as μ_P and σ_P in Chapter 4. Then, an efficient portfolio would allocate w in the market portfolio and $1-w$ in risk free asset with risk free return μ_f. Therefore its excess return is

$$R - \mu_f = wR_M + (1-w)\mu_f - \mu_f = w(R_M - \mu_f).$$

The mean and variance of the excess return is $\mu_R - \mu_f = w(\mu_R - \mu_f)$ and $\sigma_R^2 = w^2\sigma_M^2$. They all share the same slope or Sharpe's ratio:

$$\frac{\mu_M - \mu_f}{\sigma_M.} \tag{5.1}$$

Consequently,

$$\frac{\mu_R - \mu_f}{\sigma_R} = \frac{\mu_M - \mu_f}{\sigma_M}.$$

Rewrite this equation as

$$\mu_R = \mu_f + \frac{\mu_M - \mu_f}{\sigma_M}\sigma_R \tag{5.2}$$

and view the right hand side as a function of the risk σ_R. This function is linear and would be a line on the (σ_R, μ_R) plane with intercept μ_f and slope $(\mu_M - \mu_f)/\sigma_M$. This produces the so-called *capital market line* (CML). Note that $w = \sigma_R/\sigma_M$.

It follows that the optimal way of investing in the market is to simply invest a percentage of the portfolio, say w, on the market portfolio, which is often represented by the index fund, and the rest $1-w$ on risk free asset, such as T-bond or money market. The value of w is chosen according to risk tolerance. Suppose one can tolerate zero risk. Then $w = 0$, meaning that the entire portfolio is on the risk free asset. If one can tolerate half of the risk of the market, invest half on the market portfolio and half on risk free asset. If one can tolerate the same risk as that of the market, invest all on the market portfolio. If one can tolerate twice of the risk of the market, invest all and borrow the same amount to invest on the market portfolio.

Let R_j be the return of the j-th security in the market. R_j and μ_f and R_M actually depend on the time t. For simplicity, we have suppressed the index t. One can construct a linear regression model to relate the excess return of security j with that of the market:

$$R_j - \mu_f = \alpha_j + \beta_j(R_M - \mu_f) + \epsilon_j \tag{5.3}$$

where ϵ_j, $j = 1, ..., N$ are assumed to be independent identically distributed following $N(0, \sigma_{\epsilon,j}^2)$, and independent of R_M.

From the above regression model, we see that the random variation of R_j could only come from two sources: the market variation of R_M and the variation of ϵ_j. With variance decomposition, we have

$$\text{var}(R_j) = \beta_j^2 \text{var}(R_M) + \text{var}(\epsilon_j) = \beta_j^2 \sigma_M^2 + \sigma_{\epsilon,j}^2. \tag{5.4}$$

The first term on the right side measures the systematic risk due to the impact of the market variation, while the second term measures the unique risk or idiosyncratic risk due to the company itself.

As to be shown, the CAPM implies $\alpha_j = 0$. As a result, the linear regression model becomes

$$R_j - \mu_f = \beta_j (R_M - \mu_f) + \epsilon_j \tag{5.5}$$

which is the so-called *security characteristic line* (SCL).

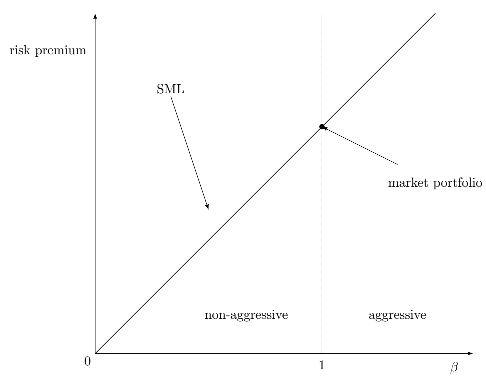

Figure 5.1

Taking expectation on both sides, if follows that

$$\mu_j - \mu_f = \beta_j (\mu_M - mu_f). \tag{5.6}$$

This is the *security market line* (SML). Taking covariance of both sides of (5.4) with R_M, we have the widely cited beta:

$$\beta_j = \frac{\text{cov}(R_j, R_M)}{\text{var}(R_M)} = \frac{\sigma_{j,M}}{\sigma_M^2}. \tag{5.7}$$

The CAPM is most commonly described, if in one single formula, by (5.6). Even it appears to rather simple, the implication is quite profound. First, (5.5) and (5.6) shows the beta describes the systmatic risk of the security, associated with the market. Low beta means the security has relatively small volatility relative to the market, aside from its own idiosyncratic risk $\sigma_{\epsilon,j}$. Based on this caliberation, beta has become one of the major characteristic of securities, or more generally portfolios. Second, (5.6) implies, the risk premium of security j is entirely proportional to the systematic risk, with the proportionality being the market risk premium. As a result, the higher excess return is totally explained or caused by higher beta describing the systematic risk.

5.2. The derivation and the risk reduction.

We show here mathematically why the CAPM, as shown in (5.6), is true. Recall that the market portfolio is the tangency portfolio, which has the highest Sharpe ratio. Consider another portfolio which places $1 - w$ on the market portfolio and w on security j. Then the return of this portfolio is $(1 - w)R_M + wR_j$. Its mean and variance of the return are, respectively, $(1 - w)\mu_M + w\mu_j$ and $v(w) = (1 - w)^2\sigma_M^2 + 2w(1 - w)\sigma_{j,M} + w^2\sigma_j^2$. Then, its Sharpe ratio is

$$S(w) = \frac{(1 - w)\mu_M + w\mu_j - \mu_f}{v(w)^{1/2}} = \frac{(1 - w)\mu_M + w\mu_j - \mu_f}{((1 - w)^2\sigma_M^2 + 2w(1 - w)\sigma_{j,M} + w^2\sigma_j^2)^{1/2}}.$$

The $S(w)$ as a function of w achieves the maximum when $w = 0$, since the tangency portfolio has the highest Sharpe ratio among all portfolios. This imples,

$$\dot{S}(w)|_{w=0} = 0, \tag{5.8}$$

where $\dot{S}$ is the derivative function of S. With some simple algebra, we have

$$\dot{S}(w) = (-\mu_M + \mu_j)v(w)^{-1/2} - v(w)^{-3/2}((1-w)\mu_M + w\mu_j - \mu_f)((w-1)\sigma_M^2 + (1-2w)\sigma_{j,M} + w\sigma_j^2)$$

Plug $w = 0$ into the above expression and notice that $v(0) = \sigma_M^2$, it follows from (5.8) that

$$\dot{S}(0) = (-\mu_M + \mu_j)/\sigma_M - (\mu_M - \mu_f)(\sigma_{j,M} - \sigma_M^2)/\sigma_M^3 = 0,$$

which gives the SML (5.6). The proof validates the zero-intercept claim of CAPM in the SCL (5.5). An illustration may be seen in the Figure 5.2.

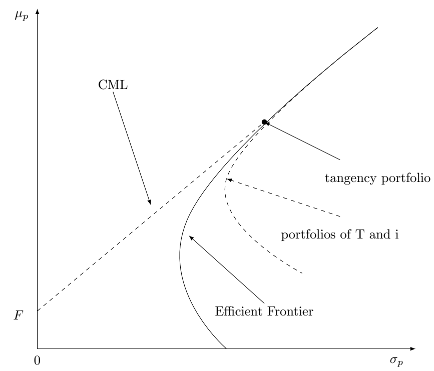

Figure 5.2

The market portfolio itself has beta equal to 1, which serves as a benchmark. Any security or portfolio with beta greater than 1 (less than 1) can be regarded as aggressive (non-aggressive). Consider an arbitrary portfolio with weights $(w_0, \mathbf{w})$ on risk free asset and the rest p risky assets on the market. Let $\beta = (\beta_1, ..., \beta_p)^T$ denote the beta for all securities. Then, the beta of this portfolio is

$$\mathbf{w}^T\beta = \mathbf{w}^T\Sigma\mathbf{w}^\star/\mathbf{w}^\star\Sigma\mathbf{w}^\star = \mathbf{w}^T\Sigma\mathbf{w}^\star/\sigma_M^2,$$

where $\mathbf{w}^\star$ is the weights of the tangency portfolio and Σ is the variance matrix of the returns of all securities in the market. And the mean excess return of this portfoio is still the product of its beta and the mean market excess return.

In summary, CAPM claims all securities risk premiums, the mean excess returns, are proportional to its systematic risk to the market, with the proportionality being the market risk premium. In other words, higer excess returns are results of taking higher systematic risk.

From the variance decomposition (5.4), one can construct portfolios in attempt to reduce the unique risks, which according to the CAPM, is irrelevant with the mean returns of the portfolio. Assume, for example, the cross sectional errors ϵ_j in (5.5) are uncorrelated with equal variance, say σ_ϵ^2, (A huge assumption). Then, the portfolio with weights $(w_0, \mathbf{w})$ would have mean excess returns $\mathbf{w}^T\beta(\mu_M - \mu_f)$ and variance, under the above ideal assumption,

$$(\mathbf{w}^T\beta)^2\sigma_M^2 + \sigma_\epsilon^2\|\mathbf{w}\|^2.$$

The second term is the unique risk. By triangle inequality,

$$\|\mathbf{w}\|^2 = \sum_{j=1}^p w_j^2 \geq (1/p)\sum_{j=1}^p |w_j| \geq 1/p(1-w_0)$$

with the equality hold when $w_1 = ... = w_p = (1 - w_0)/p$.

5.3. Statistical issues.

The theoretical expression of the beta in (5.7) gives a sample analgue:

$$\hat{\beta}_j = \frac{\sum_{t=1}^{n} R_{j,t} - \bar{R}_j)(R_{M,t} - \bar{R}_t)}{\sum_{t=1}^{n} R_{M,t} - \bar{R}_M)^2}$$

as an estimator of β_j. Here $t = 1, ..., n$ refer to a recent past period. With some further assumption, the asymptotic normal distribution of $\hat{\beta}_j - \beta_j$ can be obtained.

A more important statistical question is testing whether the CAPM is correct, or largely correct. Since the CAPM implies $\alpha_j = 0$ in the regression model, one can appeal to a standard hypothesis testing

$$\begin{cases} H_0 : \alpha_j = 0 \\ H_a : \alpha_j \neq 0 \end{cases}$$

We know the least squares estimation in Chapter 2 that, $\hat{\alpha}_j = \bar{R}_j - \hat{\beta}_j \bar{R}_M$. The t-based tests are easily available. By running linear regression model in R or any other software, the output would show the t-statistic and the p-value for the intercept. The smaller the p-value, the more evidence against the CAPM.

The above test was for an individual security. A more subtle problem is investigating the CAPM for a large number of securities and collectively assessing the evidence for/against the CAPM. This is a problem of testing a large number of hypothesis, that may involve sophisticated statistical techniques.

In the theory of CAPM, the alphas should be zero. In reality, the alpahs may not be zero. Securities with positive alphas are interpreted as being underpriced in the past, resulting in abnormal positive return. Likewise, those with negative alphas are interpreted as being overpriced in the past.

5.4. The model assumptions.

There exist extensive evidence in financial markets in favor of the CAPM. However, there also exist critics and conflicting evidence. The main underlying assumptions of CAPM are related with the widely know and yet also controversial efficient market hypothesis (EMH). These assumptions include:

(1). All market participants are rational, risk averse, rational, are broadly diversified, seek to maximise their own utility.

(2). Perfect information flow freely available to all market participants at the same time. Consequently all market participants have the same expectations

(3). No liquidity or transaction related restrictions, and all can long and short any security and the risk free asset.

The CAPM can be viewed as results of an idealized financial market rather than real world market. Some mpirical studies show that low beta stocks may offer higher returns than the model would predict, contradicting the CAPM. From a purely technical point of view, one can see that the derivation relies on the market portfolio or tangency portfolio, which is theoretically clear but practically can at best be approximated, with varying degree of accuracy. The popular market indices are often widely used as an proxies for this tangency portfolio,

In spite of the increasing critics in recent years, the CAPM has several advantages over other methods, explaining why it has remained popular for half a century. Before something better appears, the CAPM remains one of most widely used financial models in theory and in practice.

5.5. Fama-French three factor and five factor models.

The CAPM relates the returns of a security or a portfolio with only one variable, the return of the market. The variables shall be called factors here. Empirical studies show that there exist pricing anamolies in relation with the size of the companies and the value of the securites, that cannot be explained properly by the CAPM. The Fama-French (Fama and French, 1993, 1995) three factor model can be viewed as an extension/ramification of the CAPM by using two additional factors:

$$R_j - \mu_f = \beta_{j0} + \beta_{j1}(R_M - \mu_f) + \beta_{j2}\text{SMB} + \beta_{j3}\text{HML} + \epsilon_j,$$

where R_j is the return of security j, μ_f is the risk free return, R_M is the return on the value-weight market portfolio, SMB (standing for small minus big) is the return on a diversified portfolio of small stocks minus the return on a diversified portfolio of big stocks, HML (standing for high minus low) is the difference between the returns on diversified portfolios of high and low book-to-market stocks, and ϵ_j is a zero-mean error term. Suppose the factor exposure parameters (not their estimates) β_{j1}, β_{j2} and β_{j3} capture all variation in expected returns. Then the intercept β_{j0} is zero for all securities and portfolios j. Note that the time-dependent subindex t is supressed from R_j, R_M, SMB , HML and ϵ_j.

Fama and French (2015) extended their three factor model to five-factor model by adding a further two factors – profitability and investment:

$$R_j - \mu_f = \beta_{j0} + \beta_{j1}(R_M - \mu_f) + \beta_{j2}\text{SMB} + \beta_{j3}\text{HML} + \beta_{j4}\text{RMW} + \beta_{j5}\text{CMA} + \epsilon_j,$$

where RMW is the difference between the returns on diversified portfolios of stocks with robust and weak profitability, and CMA is the difference between the returns on diversified portfolios of the stocks of low and high investment firms, which we call conservative and aggressive. Likewise, if the exposures to the five factors, $\beta_{j1}, ..., \beta_{j5}$ capture all variation in expected returns, the intercept β_{j0} in the above equation would be zero for all securities and portfolios j.

5.6. Examples.

Example 1.
We use daily, weekly, monthly returns of 100 stocks (in Table 1, Chapter 4) in China market and the Shanghai Composite Index from Jan 1, 2001, to Oct 10, 2016. The Shanghai Composite Index was taken as the market returns. We use 2% as the risk-free asset returns. The excess returns are returns of 100 stocks minus the risk free returns. We are interested in testing whether the CAPM is true in China market.

By linear regression and hypothesis test for the

$$H_0 : \alpha_i = 0, \quad 1 \le i \le 100$$

$$H_a : \text{ otherwise.}$$

we get 100 p-values for each sort of return respectively, say, daily, weekly, monthly. The histograms and densities of these p-values are shown in Figure 1. It is seen that most of the p-values are greater than significant level $\alpha = 0.05$. Hence we conjecture that the intercept terms α_i is zero. To verify the conclusion, for these multiple comparisons, we use the Bonferroni correction, Benjamini-Hochberg procedures separately to make corrections to original p-values. The p-values and adjusted p-values are plotted in Figure 2. The three plots with respect to daily, weekly and monthly are very similar. Since all the adjusted p-values are above level 0.05, we do not reject null hypothesis. It turns out that the CAPM holds for China market at level 0.05.

Exercises.

5.1. Assume that the risk free return is 5% and the mean and variance of the return of the market portfolio is 10% and 0.002. What is the beta of a portfolio with mean return 20%? What is the covariance between the returns of this portfolio and of the market portfolio?

5.2. Assume the cross sectional errors are mean zero with a variance matrix Σ_ϵ, which is a p by p matrix. Assume $w_0 = 0$, compute the optimal $\mathbf{w}$ in minimizing the unique risk of the portfolio.

5.3. Explain why (5.6) and (5.7) also hold for portfolios.

5.4. Download daily data of the Hong Kong Stock Exchange, from Yahoo finance or from Bloomberg. And investigate to find evidence, either in favor of or against, the CAPM or the Fama-French models.

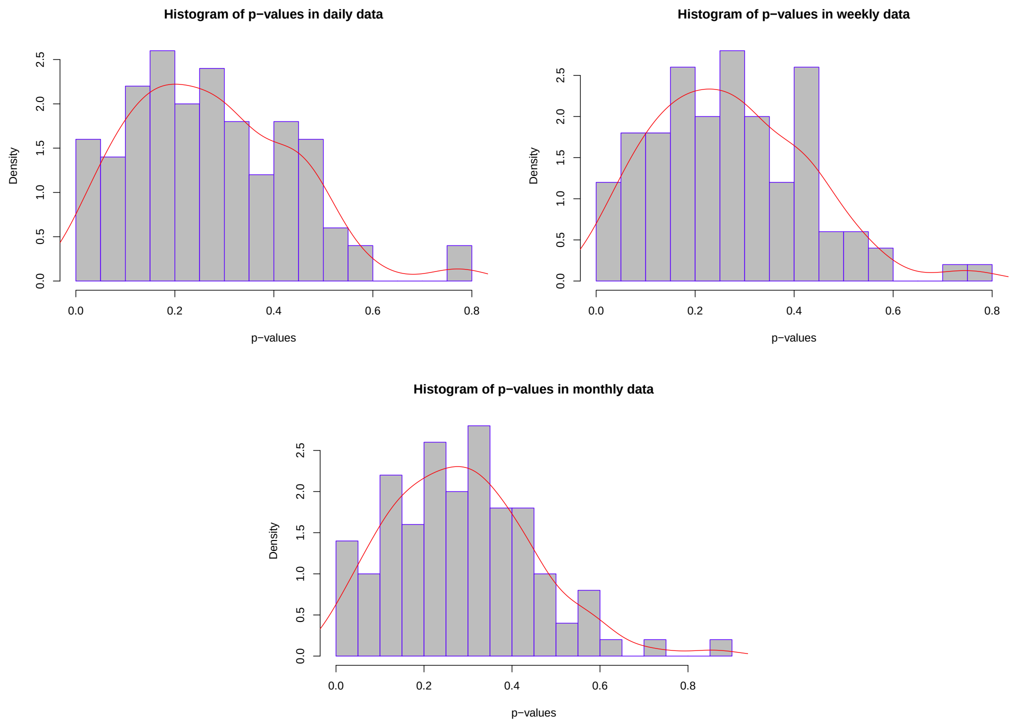

Figure 1: Histogram density of p-values in daily, weekly, monthly data. The red lines are the fitted smoothy densities.

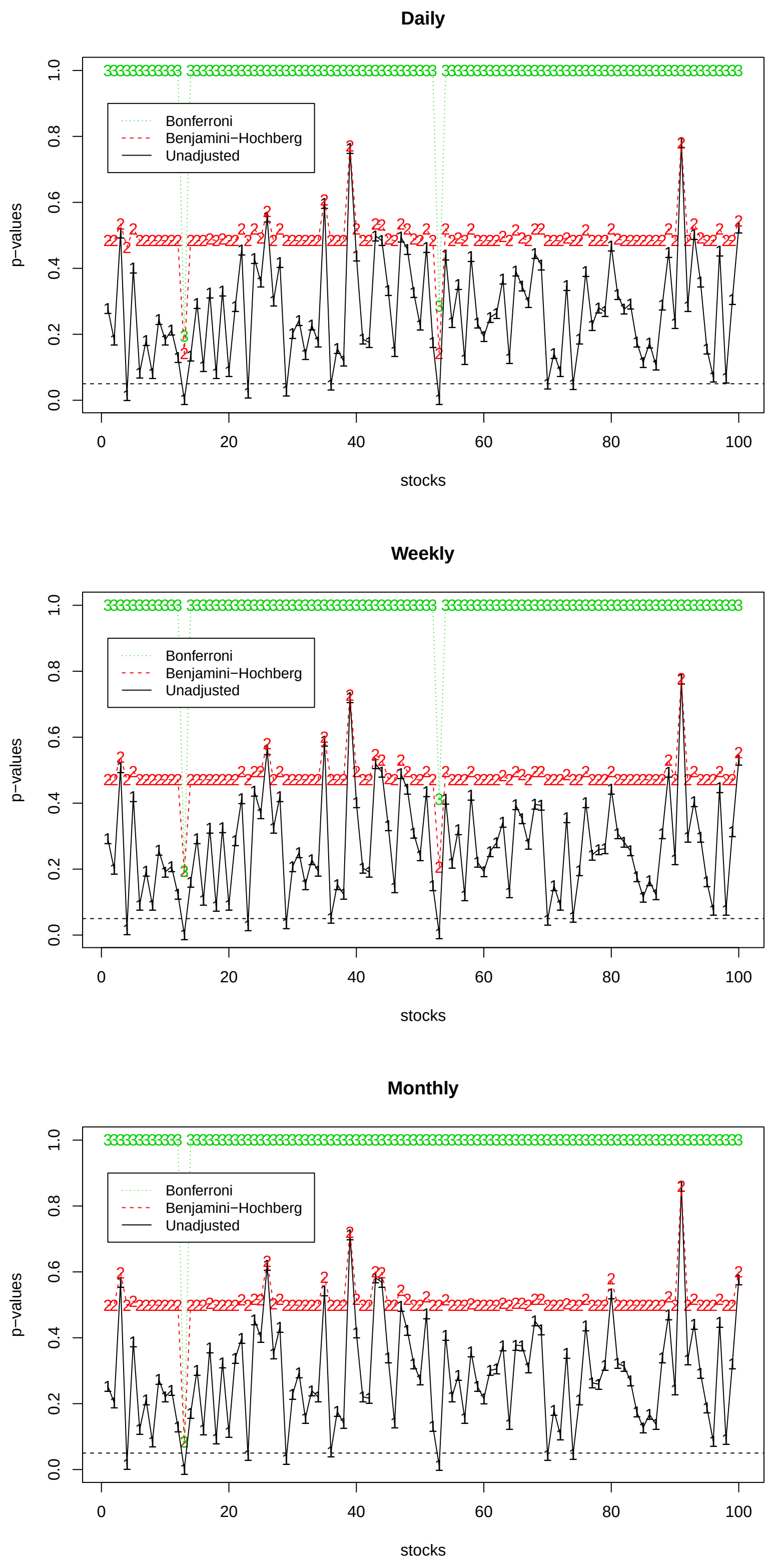

Figure 2: plot of p-values, Bonferroni adjusted p-values, Benjamini-Hochberg adjusted p-values in daily, weekly, monthly data. The dashed marked the $\alpha = 0.05$