7.3. The asymptotic behavior of renewal process.

\(X_1, X_2, \ldots\) are iid positive inter-occurrence times with mean \(\mu\), variance \(\sigma^2\) and cumulative distribution function (cdf) \(F\). \(W_n = \sum_{i=1}^{n} X_i, n \geq 1\) are waiting times. \((W_0 = 0)\). And \(N(t) = \max\{n : W_n \leq t\}\) is the renewal process over time \(t\), discrete or continuous. \(M(t) = E(N(t))\) is the renewal function.

7.3.1. The elementary renewal theorem

\[
\lim_{t \to \infty} M(t)/t = 1/\mu
\]

We may understand the limit theorem naively from the renewal theorem that we have proved:

\[
E(W_{N(t)+1}) = [E(N(t)) + 1]E(X_1) = M(t)\mu + \mu
\]  

(7.4)

Observe that \(W_{N(t)+1} = t + \gamma_t\). Therefore \(t + E(\gamma_t) = M(t)\mu + \mu\). Hence

\[
M(t)/t = 1/\mu + \frac{E(\gamma_t) - \mu}{t\mu}
\]  

(7.5)

In other words, \(W_{N(t)+1}\) is beyond time \(t\) by the residual life time \(\gamma_t\), which is supposedly at the order of 1 (in other words, it does not go to infinity). Then (7.4) follows.

7.3.2. Refined renewal theorem

Assume the cdf of the inter-occurrence times \(F\) is continuous. Then,

\[
\lim_{t \to \infty} \left[ M(t) - \frac{t}{\mu} \right] = \frac{\sigma^2 - \mu^2}{2\mu^2}
\]  

(7.6)

(7.4) is equivalent to \(M(t) - t/\mu = o(t)\), while (7.6) is equivalent to \(M(t) - t/\mu = (\sigma^2 - \mu^2)/(2\mu^2) + o(1)\). It is clear that (7.6) is indeed refined over (7.4). (Guess what \(\lim_{t \to \infty} M'(t)\) is.)

7.3.3. Asymptotic behavior the renewal process \(N(\cdot)\).

\[
\frac{E(N(t))}{t} \to \frac{1}{\mu}, \quad \frac{\text{var}(N(t))}{t} \to \frac{\sigma^2}{\mu^3}, \quad \frac{N(t) - t/\mu}{\sqrt{t\sigma^2/\mu^3}} \to N(0,1),
\]  

(7.7)

as \(t \to \infty\). Here \(N(0, 1)\) is the standard normal distribution.

7.3.4. Asymptotic distributions of current life time \(\delta_t\) and residual life time \(\gamma_t\) for continuous inter-occurrence times.

Assume the cdf of the inter-occurrence times \(F\) is continuous. Then, for all \(x \geq 0\),

\[
P(\gamma_t \leq x) \to \frac{1}{\mu} \int_0^x (1 - F(s)) ds
\]  

(7.8)

This provides the asymptotic distribution for the residual life time \(\gamma_t\). Based this result, we can derive the limiting joint distribution of \(\gamma_t\) and \(\delta_t\) as follows. For any \(x \geq 0, y \geq 0\), observe that

\[
\{\gamma_t > x\} = \{N(t + x) - N(t) = 0\}, \quad \{\delta_t > y\} = \{N(t) - N(t - y) = 0\}
\]  

(7.9)

For convenience, let \(N(s) = 0\) for \(s < 0\). Then,

\[
P(\delta_t > y, \gamma_t > x) = P(N(t) - N(t - y) = 0, N(t + x) - N(t) = 0)
\]  

by (7.9)

\[
= P(N(t + x) - N(t - y) = 0)
\]  

by (7.9) again

\[
\to 1 - \frac{1}{\mu} \int_{x+y}^{x+y} (1 - F(s)) ds
\]  

by (7.8)

\[
= \frac{1}{\mu} \int_{x+y}^{\infty} (1 - F(s)) ds
\]
By letting \( x = 0 \), we have

\[
P(\delta_t > y) \to \frac{1}{\mu} \int_0^\infty (1 - F(s))ds \leftarrow P(\gamma_t > y)
\]

It is seen that \( \delta_t \) and \( \gamma_t \) have the same asymptotic distribution.

Naively, we could calculate

\[
E(\gamma_t) \to \int_0^\infty \frac{1}{\mu} (1 - F(x))dx \quad \text{This limit is not mathematically rigorous}
\]

\[
= \frac{1}{2\mu} \int_0^\infty (1 - F(x))dx^2 = -\frac{1}{2\mu} \int_0^\infty x^2d(1 - F(x))
\]

\[
= \frac{1}{2\mu} \int_0^\infty x^2dF(x) = \frac{1}{2\mu}(\sigma^2 + \mu^2)
\]

Now you may turn back to (7.5) to understand the refined renewal theorem (7.6).

Remark. It can be seen that, for continuous \( X_i \),

\[
\lim_{t \to \infty} [E(\beta_t) - \mu] = \frac{\sigma^2}{\mu}
\]

\[
\lim_{t \to \infty} \frac{E(\beta_t) - \mu}{\mu} = \frac{\sigma^2}{\mu^2}. \tag{7.10'}
\]

While (7.10') represents the limit of the bias of total life time over the ordinary life time (inter-occurrence times), (7.10) represents the limit of the relative bias. The larger the variance of the inter-occurrence times \( X_i \), the more biased up-ward the total life time \( \beta_t \) over the ordinary life times.

The following are two examples illustrating the asymptotic theory.

**Example 7.4** Suppose the inter-occurrence times \( X_1, X_2, \ldots \) are iid \( \sim F \), with density \( f(x) = \theta - x\theta^2/2 \) for \( 0 \leq x \leq 2/\theta \); where \( \theta > 0 \). Then,

\[
F(x) = \int_0^x f(s)ds = \theta x - x^2\theta^2/4; \quad 0 \leq x \leq 2/\theta.
\]

And \( \mu = E(X_1) = 2/(3\theta) \) and \( \sigma^2 = 2/(9\theta^2) \).

\( N(t) \) is a renewal process based on \( X_1, X_2, \ldots \). Then, as \( t \to \infty \),

(1). (Elementary renewal theorem)

\[
M(t)/t \to 1/\mu = 3\theta/2;
\]

(2). (Refined renewal theorem), moreover,

\[
M(t) = t/\mu + \frac{\sigma^2 - \mu^2}{2\mu^2} + o(1) = 3\theta t/2 + \frac{2/9 - 4/9}{2 \times 4/9} + o(1) = 3\theta t/2 - 1/4 + o(1);
\]

(3). (Central limit theorem for the renewal process)

\[
\frac{N(t) - t/\mu}{\sqrt{t\sigma^2/\mu^3}} = \frac{N(t) - 3\theta t/2}{\sqrt{3t\theta/4}} \to N(0,1)
\]

(4). (Asymptotic distribution for \( \gamma_t \) and \( \delta_t \))

\[
P(\gamma_t < x) \to \frac{1}{\mu} \int_0^x (1 - F(s))ds = \frac{3\theta}{2} \int_x^\infty (1 - \theta s + \frac{x^2\theta^2}{4})ds = \frac{3\theta}{2}\{x - \theta x^2/2 - \theta^2 x^3/12\}
\]
for $0 \leq x \leq 2/\theta$. As we know, $\delta_t$ has the same limiting distribution as that of $\gamma_t$. Furthermore,

$$E(\beta_t)/\mu \to 1 + \frac{\sigma^2}{\mu^2} \to 1 + \frac{2/(9\theta^2)}{4/(9\theta^2)} = 3/2.$$ 

This limit does not have to do with $\theta$. The same is true when $F(\cdot)$ is the uniform distribution on $[0, \theta]$. However, if $F(\cdot)$ is the uniform distribution on $[\theta, \theta + 1]$ for $\theta$, the smaller the upward bias of the total life time relative to the ordinary life time.

**Example 7.5 (Renewal Process in Relation with MCs)** Suppose $Y_t; t = 0, 1, 2, \ldots$ is a Markov chain with state space being $\{0, 1\}$ and transition matrix:

$$P = \begin{pmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{pmatrix} = \begin{pmatrix} 0.4 & 0.6 \\ 0.7 & 0.3 \end{pmatrix}.$$ 

$Y_0 = 0$. The limiting probabilities $\pi_0, \pi_1$ satisfy

$$\begin{cases} (\pi_0, \pi_1) = (\pi_0, \pi_1)P \\ \pi_0 + \pi_1 = 1 \end{cases}$$

And the solutions are $\pi_0 = 7/13$ and $\pi_1 = 6/13$. Then,

$$\lim_{t \to \infty} P(Y_t = 0) = 7/13 \quad \text{and} \quad \lim_{t \to \infty} P(Y_t = 1) = 6/13.$$ 

Moreover, our interpretation of the long run behavior of Markov chains also implies that, as $n \to \infty$,

$$\frac{\# \{i \leq n : Y_i = 0\}}{n} \to \lim_{t \to \infty} P(Y_t = 0) = 7/13. \quad (7.11)$$

In addition,

$$\frac{\# \{i \leq n, Y_i = 0, Y_{i-1} = 1\}}{n} \to \lim_{t \to \infty} P(Y_{t-1} = 1, Y_t = 0) = P_{10} \lim_{t \to \infty} P(Y_{t-1} = 1) = 0.7\pi_1 = 21/65.$$ 

(1). Set $N(t) = \# \{0 < i \leq t, Y_i = 0\}$ for $t \geq 1$ and $N(0) = 0$. Then, $N(t)$ is the total number of times the Markov Chain $\{Y_1, \ldots, Y_t\}$ stays at state 0. The above interpretation implies

$$N(t)/t \to 7/13.$$ 

We now try to approach a similar result from the perspective of a renewal process. First, we verify that $N(\cdot)$ is indeed a renewal process. Set $W_0 = 0$, and $W_1 = \min\{t > 0 : Y_t = 0\}$, $W_2 = \min\{t > W_1 : Y_t = 0\}, \ldots$ $W_{k+1} = \min\{t > W_k : Y_t = 0\}$. And let $X_k = W_k - W_{k-1}$ for $k = 1, 2, \ldots$. Then, $W_k$ is the $k$-th time that the Markov chain $Y_t$ visits state 0, counting from time 1 (not time 0). And $X_k$ is, certainly, the time period between the $k$-1-th and $k$-th visits of state 0. With some technicalities it can be shown $X_k$ are iid positive random variables with distribution:

$$P(X_k = i) = \begin{cases} 0.4 & i = 1 \\ 0.6 \times 0.7 & i = 2 \\ \vdots \\ 0.6 \times 0.3^{n-2} \times 0.7 & i = n \quad \text{for} \quad n \geq 3 \end{cases}$$

And $\mu = E(X_k) = 13/7$. By renewal theorem, we have

$$\frac{E(N(t))/t}{1/\mu} = \frac{N(t)/t - 7/13}{\sqrt{\sigma^2/(\mu^2)}} \to N(0, 1)$$

which implies (7.11)
(2). Set

\[ N(t) = \sum_{i=1}^{t} I(Y_i = 0, Y_{i-1} = 1). \]

Then \( N(t) \) is the number of times the Markov chain \( \{Y_i\} \) returns to state 0 from state 1 over times \( \{1, 2, \ldots, t\} \). To understand why \( N(\cdot) \) is a renewal process. We set \( W_0 = 0 \).

\[ W_1 = \min\{k \geq 2 : Y_k = 0, Y_{k-1} = 1\}, \]

\[ W_2 = \min\{k > W_1 : Y_k = 0, Y_{k-1} = 1\}, \]

\[ \ldots \]

\[ W_n = W_n - W_{n-1} \]

for \( n \geq 1 \). Then \( X_n = W_n - W_{n-1} \) are iid with

\[ P(X_n = k) = \sum_{i=0}^{k-2} 0.4^i \times 0.6 \times 0.3^{k-2-i} \times 0.7, \quad k \geq 2. \]

And

\[ \mu = E(X_n) = 65/21. \]

By the renewal theorem, we have, as \( t \to \infty \),

\[ E(N(t))/t \to 1/\mu = 21/65. \]

\[ \frac{N(t)/t - 21/65}{\sqrt{\sigma^2/(t\mu^3)}} \to N(0, 1), \]

which implies (7.12).

**Exercise 7.5** Suppose the inter-occurrence times \( X_1, X_2, \ldots \), are iid \( \sim Unif[0, 1] \). State (1). elementary renewal theorem; (2). the refined renewal theorem; (3). the central limit theorem for the renewal process; and (4). \( \lim_{t \to \infty} P(\gamma_t > x, \delta_t > y) \) for all \( 0 < x, y < 1 \). (5). \( \lim_{t \to \infty} P(\beta_t < x) \) for \( 0 < x < 1 \); and (6). \( \lim_{t \to \infty} E(\beta_t) \).

**Exercise 7.6** Suppose the inter-occurrence times \( X_1, X_2, \ldots \), are iid following a common uniform distribution on \( \{1, 2, 3, 4, 5\} \). Find the limit of \( P(N(t) > t/2 + \sqrt{t}) \) as \( t \to \infty \).

**Exercise 7.7** Suppose the inter-occurrence times \( X_1, X_2, \ldots \), are iid with a common continuous density \( f(\cdot) \) on \( [0, \infty) \). Suppose \( \gamma_t \) and \( \delta_t \) are asymptotically independent, i.e., \( \lim_{t \to \infty} P(\gamma_t > x, \delta_t > y) = \lim_{t \to \infty} P(\gamma_t > x)P(\delta_t > y) \) for all \( x, y \geq 0 \). Then, the renewal process must be a Poisson process.