§ 1.5. Weak law of large numbers.

For a sequence of independent r.v.s $X_1, X_2, ...$, classical law of large numbers is typically about the convergence of partial sums

$$\frac{S_n - E(S_n)}{n} = \frac{\sum_{i=1}^n [X_i - E(X_i)]}{n},$$

where $S_n = \sum_{i=1}^n X_i$ here and throughout this Chapter. A more general form is the convergence of

$$\frac{S_n - a_n}{b_n}$$

for some constants a_n and b_n. Weak law is convergence in probability and strong law is convergence a.s..

(i). *Weak law of large numbers.*

The following proposition may be called L^2 weak law of large numbers which implies the weak law of large numbers.

PROPOSITION *Suppose $X_1, ..., X_n, ...$ are iid with mean μ and finite variance σ^2. Then,*

$$S_n/n \to \mu \qquad \textit{in probability and in } L^2.$$

Proof. Write

$$E(S_n/n - \mu)^2 = (1/n)\sigma^2 \to 0.$$

Therefore L^2 convergence holds. And convergence in probability is implied by the Chebyshev inequality. □

The above proposition implies that classical weak law of large numbers holds quite trivially in a standard setup with the r.v.s being iid with finite variance. In fact, in such a standard setup strong law of large numbers also holds, as to be shown in Section 1.7. However, the fact that convergence in probability is implied in L^2 convergence plays a central role is establishing weak law of large numbers. For a example, a straightforward extension of the above proposition can be:

For independent r.v.s $X_1, ...,$, $(S_n - E(S_n))/b_n \to 0$ in probability if $(1/b_n^2)\sum_{i=1}^n \text{var}(X_i) \to 0$, for some $b_n \uparrow \infty$.

The following theorem about general weak law of large numbers is a combination of the above extension and the technique of truncation.

Theorem 1.3. WEAK LAW OF LARGE NUMBERS *Suppose $X_1, X_2, ...$ are independent. Assume*

(1). $\sum_{i=1}^n P(|X_i| > b_n) \to 0$,

(2). $b_n^{-2}\sum_{i=1}^n E(X_i^2 1_{\{|X_i| \leq b_n\}}) \to 0$,

where $0 < b_n \uparrow \infty$. Then $(S_n - a_n)/b_n \to 0$ in probability, where $a_n = \sum_{j=1}^n E(X_i 1_{\{|X_i| \leq b_n\}})$.

Proof. Let $Y_j = X_j 1_{\{|X_j| \leq b_n\}}$. Consider

$$\frac{\sum_{j=1}^n Y_j - a_n}{b_n} = \frac{\sum_{j=1}^n [Y_j - E(Y_j)]}{b_n},$$

which is mean 0 and converges to 0 in L^2 by (2). Therefore it also converges to 0 in probability. Notice that

$$P\Big(\frac{S_n - a_n}{b_n} = \frac{\sum_{j=1}^n Y_j - a_n}{b_n}\Big) = P(S_n = \sum_{j=1}^n Y_j)$$

$$
\begin{aligned}
&\geq \quad P(X_j = Y_j \ \text{ for all } \ 1 \leq j \leq n) = \prod_{j=1}^{n} P(X_j = Y_j) \qquad \text{by independence} \\
&= \quad \prod_{j=1}^{n} P(|X_j| \leq b_n) = \prod_{j=1}^{n} [1 - P(|X_j| > b_n)] = e^{\sum_{j=1}^{n} \log[1-P(|X_j|>b_n)]} \\
&\approx \quad e^{-\sum_{j=1}^{n} P(|X_j|>b_n)} \\
&\to \quad 1 \qquad \text{by (1).}
\end{aligned}
$$

Hence $(S_n - a_n)/b_n \to 0$ in probability. □

Theorem 1.4. *Suppose $X, X_1, X_2, ...$ are iid. Then, $S_n/n - \mu_n \to 0$ in probability for some μ_n, if and only if*

$$xP(|X_1| > x) \to 0 \qquad \text{as } x \to \infty.$$

in which case $\mu_n = E(X1_{\{|X| \leq n\}}) + o(1)$.

Proof. "$\Longleftarrow$" Let $a_n = n\mu_n$ and $b_n = n$ in Theorem 1.3. Condition (1) follows. To check Condition (2), write, as $n \to \infty$,

$$
\begin{aligned}
& b_n^{-2} \sum_{i=1}^{n} E(X_i^2 1_{\{|X_i| \leq b_n\}}) = \frac{1}{n} E(X^2 1_{\{|X| \leq n\}}) \leq \frac{1}{n} E(\min(|X|, n)^2) \\
= \quad & \frac{1}{n} \int_0^\infty 2xP(\min(|X|, n) > x) dx = \frac{1}{n} \int_0^n 2xP(|X| > x) dx \\
= \quad & \frac{1}{n} \int_M^n 2xP(|X| > x) dx + o(1) \qquad \text{for any fixed } M > 0 \\
= \quad & \frac{2}{n} \int_M^n xP(|X| > x) dx + o(1) \leq 2 \sup_{x \geq M} xP(|X| > x) + o(1),
\end{aligned}
$$

as $n \to \infty$. Since M is arbitray, Condition (2) holds. And the WLLN follows from Theorem 1.3.

"$\Longrightarrow$" Let $X^*, X_1^*, ...$ be iid following the same distribution of X and are independent of $X, X_1, ...$ Set $\xi_i = X_i - X_i^*$ (symmetrization) and $\tilde{S}_n = \sum_{i=1}^{n} \xi_i$. Then, $\tilde{S}_n/n \to 0$ in probability. The Levy inequality in Exercise 1.13 implies $\max\{|\tilde{S}_j| : 1 \leq j \leq n\}/n \to 0$ in probability, which further ensures $\max\{|\xi_j| : 1 \leq j \leq n\}/n \to 0$ in probability. For any $\epsilon > 0$,

$$
\begin{aligned}
& nP(|X| \geq n\epsilon)P(|X^*| \leq .5n\epsilon) = nP(|X| \geq n\epsilon, \, |X^*| \leq .5n\epsilon) \leq nP(|X - X^*| \geq .5n\epsilon) \\
\approx \quad & 1 - [1 - P(|X - X^*| \geq .5n\epsilon)]^n = P(\max_{1 \leq j \leq n} |\xi_j| > .5n\epsilon) \to 0.
\end{aligned}
$$

As a result, for any $\epsilon > 0$,

$$nP(|X| \geq n\epsilon) \approx nP(|X| \geq n\epsilon)[1 - P(|X| \geq .5n\epsilon)] \to 0,$$

which is equivalent to $xP(|X| > x) \to 0$ as $x \to \infty$. □

Example 1.4. Suppose $X_1, X_2, ...$ are i.i.d. with common density f symmetric about 0 and c.d.f such that $1 - F(t) = 1/(t \log t)$, for $t > 3$. Then, $S_n/n \to 0$ in probability. But $S_n/n \not\to 0$, a.s..

The convergence in probability is a consequence of Theorem 1.4 with $\mu_n = 0$ and checking the condition $xP(|X| > x) \to 0$ as $x \to \infty$. The convergence a.s. is untrue because $X_n/n \not\to 0$ a.s. by Borel-Cantelli lemma. □

Corollary. *Suppose $X_1, ..., X_n, ...$ are i.i.d. with $E(|X_i|) < \infty$. Then, $S_n/n \to E(X_1)$ in probability.*

Proof. Since, as $x \to \infty$,

$$xP(|X_i| > x) = o(1)\int_0^x P(|X_i| > t)dt = o(1)\int_0^\infty P(|X_i| > t)dt = o(1)E(|X_i|),$$

the WLLN follows from Theorem 1.4. □

Example 1.5. The St. Petersberg Paradox. Let $X, X_1, ..., X_n, ...$ be iid with $P(X = 2^k) = 2^{-k}$, $k = 1, 2, ...$ Then, $E(X) = \infty$ and

$$\frac{S_n}{n \log n} \to \frac{1}{\log 2} \qquad \text{in probability.}$$

Proof. Notice that $P(X \geq 2^k) = 2^{-k+1}$. Let $k_n \approx \log\log n/\log 2$, $m_n = \log n/\log 2 + k_n$ and $b_n = 2^{m_n} = 2^{k_n} n \approx n \log n$. m_n is an integer. Then,

$$nP(X \geq b_n) = n2^{-m_n+1} \approx 2n/n \cdot 2^{-k_n} \to 0.$$

And

$$E(X^2 1_{\{|X| \leq b_n\}}) = \sum_{k=1}^{m_n} 2^{2k}2^{-k} = \sum_{k=1}^{m_n} 2^k \leq 2 \times 2^{m_n} = 2b_n.$$

Then,

$$\frac{nE(X^2 1_{\{|X| \leq b_n\}})}{b_n^2} \leq \frac{2nb_n}{b_n^2} = \frac{2n}{b_n} = \frac{2n}{2^{m_n}} = \frac{2n}{n2^{k_n}} \to 0.$$

Let $a_n = nE(X1_{\{|X| \leq b_n\}})$.

$$a_n = n\sum_{k=1}^{m_n} 2^k 2^{-k} = nm_n = n\log n/\log 2 + nk_n \approx b_n \log 2.$$

The desired convergence is implied by Theorem 1.4. □

Example 1.6. "Unfair fair game". You pay one dollar to buy a lottery. The lottery has infinite number of numbered balls. If number k occurs, you are paid by 2^k dollars. The number k ball occurs with probability

$$p_k \equiv \frac{1}{2^k k(k+1)}.$$

Is this a fair game?

In a sense, it is fair. Let X be gain/loss of the outcome. Then $P(X = 2^k - 1) = p_k, k = 1, 2, ...$ and $P(X = -1) = 1 - \sum_k p_k$. Then $E(X) = 0$.

If one buys the lottery on daily basis, one time every day. Let X_n be gain/loss of day n and S_n be the cumulative gain/loss up to day n. Then,

$$\frac{S_n}{n/\log n} \to -\log 2 \qquad \text{in probability,}$$

meaning that in the long time, he/she is nearly certainly in red. □

Example 1.7. Compute the limit of

$$\int_0^1 \cdots \int_0^1 \frac{x_1^2 + \cdots x_1^2 +}{x_1 + \cdots + x_n} dx_1 \cdots dx_n.$$

Solution. The above integral is the same as

$$E\Big(\frac{X_1^2 + \cdots + X_n^2}{X_1 + \cdots + X_n}\Big),$$

where $X_1, ..., X_n, ...$ are iid $\sim Unif[0,1]$. Since, by the WLLN

$$(1/n)\sum_{i=1} X_i^2 \to E(X_1^2) = \int_0^1 x^2 dx = 1/3 \quad \text{and} \quad (1/n)\sum_{i=1} X_i \to E(X_1) = 1/2,$$

with the convergence being convergence in probability, we have

$$\frac{X_1^2 + \cdots + X_n^2}{X_1 + \cdots + X_n} \to 2/3 \qquad \text{in probability.}$$

The r.v. on the left hand side is bounded by 1. By the dominated convergence, its mean also converges to 2/3. Then the limit of the integral is 2/3. □

REMARK. The following WLLN for array of r.v.s. is a slight generalization of Theorem 1.3. Suppose $X_{n,1}, ..., X_{n,n}$ are independent r.v.s. If

$$\sum_{i=1}^n P(|X_{n,i}| > b_n) \to 0 \quad \text{and} \quad (1/b_n^2)\sum_{i=1}^n E(X_{n,i}^2 1_{\{|X_{n,i}|\le b_n\}}) \to 0,$$

Then,

$$\frac{\sum_{i=1}^n X_{n,i} - a_n}{b_n} \to 0 \qquad \text{in probability}$$

where $a_n = \sum_{i=1}^n E(X_{n,i} 1_{\{|X_{n,i}|\le b_n\}})$.

DIY EXERCISES.

Exercise 1.17 (LEVY'S INEQUALITY) Suppose $X_1, X_2, ...$ are independent and symmetric about 0. Then,

$$P(\max_{1\le j\le n} |S_j| \ge \epsilon) \le 2P(|S_n| \ge \epsilon)$$

Exercise 1.18 Show $S_n/(n\log n) \to -\log 2$ in probability in Example 1.7. Hint: Choose $b_n = 2^{m_n}$ with $m_n = \{k : 2^{-k}k^{-3/2} \le 1/n\}$ and proceed as in Example 1.5.

Exercise 1.19 For Example 1.4, prove that $S_n/b_n \to 0$ in probability, if $b_n/(n/\log n) \uparrow \infty$.

Exercise 1.20 (MARCINKIEWICZ-ZYGMUND WEAK LAW OF LARGE NUMBERS) Suppose $x^p P(|X| > x) \to 0$ as $x \to \infty$ for some $0 < p < 2$. Prove that

$$\frac{S_n - nE(X 1_{\{|X|\le n^{1/p}\}})}{n^{1/p}} \to 0 \qquad \text{in probability.}$$