## § 1.6. Kolmogorov inequality and the convergence of series.

For r.v.s $X_1, X_2, ...$, convergence of series means the convergence of its partial sums $S_n = \sum_{i=1}^n X_i$, as $n \to \infty$. We shall denote the convergence of $S_n$ a.s. just as $\sum_{n=1}^\infty X_n < \infty$ a.s.. The following Kolmogorov inequality is the key to establishing a.s. convergence of series for independent r.v.s.

(i). Kolmogorov inequality.

**Theorem 1.5.** KOLMOGOROV INEQUALITY *Suppose $X_1, X_2, ..., X_n$ are independent with $E(X_i) = 0$ and $\mathrm{var}(X_i) < \infty$. $S_j = X_1 + ... + X_j$. Then,*

$$P(\max_{1\le j\le n} |S_j| \ge \epsilon) = \frac{\mathrm{var}(S_n)}{\epsilon^2}.$$

**Proof.** Let $T = \min\{j \le n : |S_j| \ge \epsilon\}$, with minimum of empty set being $\infty$, i.e., $T = \infty$ $|S_j| \le \epsilon$ for all $1 \le j \le n$. Then, $\{T \le j\}$ *or* $\{T = j\}$ *only depends on* $X_1, ..., X_j$. And, as a result,

$$\{T \ge j\} = \{T \le j-1\}^c = \{S_i \le \epsilon, 1 \le i \le j-1\}$$

only depends on $X_1, ..., X_{j-1}$ and therefore is independent of $X_j, X_{j+1}, ....$ Write

$$\begin{aligned}
& P(\max_{1\le j\le n} |S_j| \ge \epsilon) = P(T \le n) \ \le\ \epsilon^{-2} E(|S_T|^2 1_{\{T\le n\}}) \ \le\ \epsilon^{-2} E(|S_{T\wedge n}|^2) \\
= \ & \epsilon^{-2} E(|\sum_{j=1}^{T\wedge n} X_j|^2) = \epsilon^{-2} E(|\sum_{j=1}^{n} X_j 1_{\{T\ge j\}}|^2) \\
= \ & \epsilon^{-2} \Big\{ E(\sum_{j=1}^n X_j^2 1_{\{T\ge j\}}) + 2 \sum_{1\le i<j\le n}^{n} E(X_j X_i 1_{\{T\ge j\}} 1_{\{T\ge i\}} \Big\} \\
= \ & \epsilon^{-2} \Big\{ \sum_{j=1}^n E(X_j^2) P(T \ge j) + 2 \sum_{1\le i<j\le n}^{n} E(X_j) E(X_i 1_{\{T\ge j\}} 1_{\{T\ge i\}}) \Big\} \\
= \ & \epsilon^{-2} \sum_{j=1}^n E(X_j^2) P(T \ge j) + 0 \\
\le \ & \mathrm{var}(S_n)/\epsilon^2.
\end{aligned}$$

□

EXAMPLE 1.8. (Extension to continuous time process.) Suppose $\{S_t : t \in [0, \infty)\}$ is a process with increments that are independent, zero mean and finite variance. If the path of $S_t$ is right continuous, e.g.

$$P\Big(\max_{t\in[0,\tau]} |S_t| > \epsilon\Big) \le \frac{\mathrm{var}(S_\tau)}{\epsilon^2}.$$

The examples of such processes are, e.g., compensated Poisson process and Brownian Motion.

Kolmogorov's inequality will later on be seen as a special case of martingale inequality. In the proof of Kolmogorov inequality, we have used a *stopping time* $T$, which is a r.v. associated with a process $S_n$ or, more generally, a filtration, such that $T = k$ only depends on past and current values of the process: $S_1, ..., S_k$. Stopping time is one of the most important concepts and tools in martingale theory or stochastic processes.

(ii). Khintchine-Kolmogorov convergence theorem.

**Theorem 1.6.** (KHINTCHINE-KOLMOGOROV CONVERGENCE THEOREM) *Suppose $X_1, X_2, ...$ are independent with mean 0 such that $\sum_n \mathrm{var}(X_n) < \infty$. Then, $\sum_n X_n < \infty$ a.s., i.e., $S_n$ converges a.s. as well as in $L^2$ to $\sum_{n=1}^\infty X_n$.*

**Proof.** Define $A_{m,\epsilon} = \{\max_{j>m}|S_j - S_m| \le \epsilon\}$. Then, $\{\sum_{n=1}^\infty X_n < \infty\} = \cap_{\epsilon>0}\cup_m A_{m,\epsilon}$. By Kolmogorov's inequality

$$P(\max_{m<j\le n}|S_j - S_m| > \epsilon) \le \frac{\mathrm{var}(S_n - S_m)}{\epsilon^2} = \frac{1}{\epsilon^2}\sum_{i=m+1}^n \mathrm{var}(X_i) \le \frac{1}{\epsilon^2}\sum_{i=m+1}^\infty \mathrm{var}(X_i).$$

By letting $n \to \infty$ first and then $m \to \infty$, we have

$$\lim_{m\to\infty} P(\max_{j>m}|S_j - S_m| > \epsilon) \to 0.$$

Then $\lim_m P(A_{m,\epsilon}) \to 1$. So $P(\cup_{m\ge1}A_{m,\epsilon}) = 1$ for every $\epsilon > 0$. Hence,

$$P(\sum_n X_n < \infty) = P(\cap_{\epsilon>0}\cup_m A_{m,\epsilon}) = 1.$$

And a.s. convergence of $S_n$ holds. Denote the a.s. limit as $S_\infty$.

To show convergence of $S_n$ in $L^2$, write

$$\begin{aligned} & E[(S_n - S_\infty)^2] = E[(S_n - \lim_k S_k)^2] = E[\lim_k (S_n - S_k)^2] \\ \le\ & \liminf_k E[(S_n - S_k)^2] \qquad \text{by Fatou's lemma} \\ =\ & = \liminf_k \sum_{j=n}^k \mathrm{var}(X_j) = \sum_{j=n}^\infty \mathrm{var}(X_j) \end{aligned}$$

which tends to 0, as $n \to \infty$. Therefore convergence in $L^2$ holds. □

Example 1.9. Suppose $X_1, ...$ are iid with zero mean and finite variance. Then $\sum_n a_n X_n < \infty$ a.s. if and only if $\sum_n a_n^2 < \infty$.

"$\Longleftarrow$" is a direct consequence of Theorem 1.6.. "$\Longrightarrow$" follows from the central limit theorem to be shown in Chapter 2.

(iii). Kolmogorov three series theorem

For independent random variables, Kolmogorov three series theorem is the ultimate result in providing sufficient and necessary conditions for the convergence of series a.s..

**Theorem 1.7.** (Kolmogorov Three Series Theorem) *Suppose $X_1, X_2, ...$ are independent. Let $Y_n = X_n 1_{\{|X_n|\le1\}}$ Then, $\sum_n X_n < \infty$ a.s. if and only if (1). $\sum_n P(|X_n| > 1) < \infty$; (2). $\sum_n E(Y_n) < \infty$; and (3). $\sum_n \mathrm{var}(Y_n) < \infty$.*

**Proof**. "$\Longleftarrow$": The convergence of $\sum_n(Y_n - E(Y_n))$ is implied by (3) and Theorem 1.6. Together with (2), it ensures $\sum_n Y_n < \infty$ a.s.. On the other hand, Condition (1) and Borel-Cantelli lemma implies $P(X_n \ne Y_n, i.o.) = 0$. Consequently, $\sum_n X_n$ converges.

"$\Longrightarrow$" (An unconventional proof). It's straightforward that Condition (1) holds. Then $\sum_n Y_n < \infty$ a.s. since $P(X_n \ne Y_n,\ i.o.) = 1$. If condition (3) does not hold, by the central limit theorem to be shown in the next chapter,

$$\frac{1}{\sqrt{\sum_{i=1}^n \mathrm{var}(Y_i)}}\sum_{i=1}^n [Y_i - E(Y_i)] \to N(0,1),$$

in distribution. Hence $P(|\sum_{i=1}^n Y_i| > M) \to 0$ as $n \to \infty$ for any fixed $M > 0$, which contradicts with $\sum_n Y_n < \infty$ a.s.. Hence condition (3) holds. Theorem 1.6 then ensures $\sum_n(Y_n - E(Y_n)) < \infty$ a.s.. As a result, $\sum_n E(Y_n) < \infty$ and condition (2) also holds. □

Remark. Suppose $X_n$ is truncated at any constant $\epsilon > 0$ rather than 1 in Theorem 1.7, the theorem still holds.

**Corollary.** *Suppose $X, X_1, X_2, ...$ are iid with $E(|X|^p) < \infty$ for some $0 < p < 2$. Then, $\sum_{n=1}^{\infty}[X_n - E(X)]/n^{1/p} < \infty$ a.s. for $1 < p < 2$; and $\sum_{n=1}^{\infty} X_n/n^{1/p} < \infty$ a.s. for $0 < p < 1$.*

We leave the proof as Exercise 1.22.

## DIY Exercises

*Exercise 1.21.* Suppose $S_0 \equiv 0, S_1, S_2, ...$ form a square integrable martingale, i.e., for $k = 0, 1, ..., n$, $E(S_k^2) < \infty$ and $E(S_{k+1}|\mathcal{F}_k) = S_k$ where $\mathcal{F}_k$ is the $\sigma$-algebra generated by $S_1, ..., S_k$. Show that Kolmogorov's inequality still holds.

*Exercise 1.22.* Prove the Corollary following Theorem 1.6..

*Exercise 1.23.* For positive independent r.v.s $X_1, X_2, ...$, show that the following three statements are equivalent: (a). $\sum_n X_n < \infty$ a.s.; (b). $\sum_n E(X_n \wedge 1) < \infty$; (c). $\sum_n E(X_n/(1 + X_n)) < \infty$.

*Exercise 1.24.* Raise a counterexample to show that there exists $X_1, X_2, ...$ iid with $E(X) = 0$ but $\sum_n X_n/n \not< \infty$ a.s..