Final Exam, Math5411

Totally 5 problems.

Must hand-in your exam before or at 12 noon, Dec 19, to Professor Chen in Room 3426 (Phone number 23587425) and sign your name.

Name:

ID Number:

1. Suppose $X, X_1, X_2, \ldots$ are iid positive random variables and $r > 0$. Show that the following statements are equivalent.

   (i). $X_n/n^r \to 0$ almost surely.

   (ii). $E(X^{1/r}) < \infty$.

   (iii). $\sum_{n=1}^{\infty} P(X > n^r) < \infty$.

2. Let $X_1, X_2, ...,$ be iid random variables symmetric about 0. Assume $nP(|X| > c_n) \to 0$ and $nb_n^{-2}E(X^2 1_{\{|X| \le c_n\}}) \to 0$, where b_n and c_n are positive constants. Show that

$$\frac{1}{b_n} \sum_{i=1}^{n} X_i \to 0 \qquad \text{in probability.}$$

3. Suppose X_i are independent random variables such that $P(|X_i| \leq 1) = 1$ and $E(X_i) = 0$. Let a_i be positive constants such that $a_i \leq 1$ and $\sum_{i=1}^{\infty} a_i = \infty$. Prove that

$$\frac{\sum_{i=1}^{n} a_i X_i}{\sum_{i=1}^{n} a_i} \to 0, \qquad a.s..$$

4. Suppose $X, X_1, X_2 ...$, are iid random variables with mean 0.

   (i). Prove that if the distribution of X is symmetric about 0, then $\sum_{j=1}^{n} X_j/j$ converges almost surely.

   (ii). Raise an example of the distribution of X (caution: must be mean 0) such that $\sum_{j=1}^{n} X_j/j$ does not converge almost surely.

5. Suppose $X_1, ..., X_n, ...$ are iid random variables with the exponential distribution with mean 1. Let $G_n = \sum_{i=1}^{n} X_i^2$. Let $\Phi(\cdot)$ be the cumulative distribution function of the standard normal distribution. Prove that $\sqrt{n}[\exp(G_n/n) - \exp(2))]$ converges in distribution and specify the limiting distribution.