# Final Exam, Math5411

Totally 5 problems.

Must hand-in your exam before or at 12 noon, Dec 11, to Ms. Debbie Poon, and sign your name
Location: the general office of Mathematics department
Phone number: 23587425; email: madebbie@ust.hk.

Name:

ID Number:

1. Assume $\sum_{n=1}^{\infty} E(|X_n|^{p_n}) < \infty$, where $0 < p_n \leq 2$, and $E(X_n) = 0$ if $p_n > 1$. Show that $\sum_{j=1}^{n} X_j$ converges almost surely.

2. Suppose $X_j, j \geq 1$, are independent random variables with mean 0 and variance $\sigma_j^2, j \geq 1$. Let $S_n = \sum_{j=1}^n X_j$ and $s_n^2 = \sum_{j=1}^n \sigma_j^2$. Prove that, for $x > 0$,

$$P(S_n \geq x) \geq (1 - \frac{2s_n^2}{x^2}) \sum_{j=1}^n P(X_j \geq 2x)$$

3. Suppose $X, X_1, X_2...$, are iid random variables with mean 0.

(i). Prove that if the distribution of $X$ is symmetric about 0, then $\sum_{j=1}^{n} X_j/j$ converges almost surely.

(ii). Raise an example of the distribution of $X$ (caution: must be mean 0) such that $\sum_{j=1}^{n} X_j/j$ does not converge almost surely.

4. Suppose $X, X_1, X_2 ...,$ are iid random variables with mean 0 and variance 1. For a sequence of positive constants $c_n$, suppose $c_n/C_n \to 0$ where $C_n = \sqrt{\sum_{j=1}^n c_j^2}$. Prove that $\sum_{j=1}^n c_j X_j / C_n$ converges to the standard normal distribution.

5. Suppose $X_1, ..., X_n, ...$ are iid nonnegative random variables with mean 0 and variance 1. Let $S_n = \sum_{i=1}^n X_i$. Let $\Phi(\cdot)$ be the cumulative distribution function of the standard normal distribution. Prove that $\sqrt{n}[\Phi(S_n/n) - 1/2)]$ converges in distribution and specify the limiting distribution.