Solutions of Homework 1.

1. a). Roll a fair die 2 times, define the probability space $(\Omega, \mathcal{F}, P)$.

   b). Toss a fair coin infinite number of times, define the probability space $(\Omega, \mathcal{F}, P)$. (Hint: finite-dimensional probability is enough.)

   Solution. a). $\Omega = \{(i,j) : 1 \le i, j \le 6\}$, where (i,j) represents the outcome of roll is i-dots for the first roll and j-dots for the second. $\mathcal{F}$ is the set of all subsets of Ω. P is such that $P((i,j)) = 1/36$.

   b). $\Omega = \{\omega\}$, where $\omega = (w_1, w_2, ...)$ with $w_n = H$ or T, representing the n-th toss is head or tail. $\mathcal{F}$ is the σ-algebra generated by sets of finite dimensions. e.g., $\mathcal{F} = \sigma\{(a_1, a_2, ...)$ with only finite number of a_i being fixed as either H or $T\}$. P is such that $P(w_i = a_i \text{ for } i = 1, ..., n) = 2^{-n}$ where a_i is either H or T. □

2. Suppose $X \ge 0$ is a random variable in probability space $(\Omega, \mathcal{F}, P)$, and $E(X) = c$ with $0 < c < \infty$. For any set A in $\mathcal{F}$, define $P^*(A) = E(X1_A)/c$. Show that P^* is a probability measure, i.e., it satisfies Kolmogorov's axioms of probability.

   Solution. (i). For any $A \in \mathcal{F}$, $P^*(A) \ge 0$ since $X \ge 0$. Also, $P*(A) \le E(X)/c = 1$. (ii). since $1_\Omega = 1$, $P^*(\Omega) = E(X)/c = 1$. (iii). Suppose A_i are mutually exclusive. $P^*(\cup_i A_i) = E(X1_{\cup_i A_i})/c = \sum_i E(X1_{A_i})/c = \sum_i P^*(A_i)$. □

3. Suppose X is a nonnegative random variable.

   a). Show that $E(X) = \int_0^\infty P(X > t)dt$.

   b). Show that $E(X) < \infty$ iff $\sum_{n=1}^\infty P(X > n) < \infty$.

   Solution. a). Method 1.

$$E(X) = E(\int_0^\infty 1_{\{t \le X\}} dt) = \int_0^\infty E(1_{\{t \le X\}})dt = \int_0^\infty P(X \ge t)dt.$$

   Method 2. $E(X) = \int_0^\infty x dF(x) = -\int_0^\infty x d(1-F(x)) = -\lim_{c\to\infty} \int_0^c x d(1-F(x)) = \lim_{c\to\infty}[-(1-F(c))c + \int_0^c (1-F(x))dx$. If $E(X)$ is finite, then $(1-F(c))c \to 0$ as $c \to \infty$ (why?) and the desired equality holds. If $E(X) = \infty$, then $\int_0^\infty P(X > t)dt = \lim_{c\to\infty} \int_0^c (1-F(x))dx \ge \lim_{c\to\infty}[-(1-F(c))c + \int_0^c (1-F(x))dx] = \lim_{c\to\infty} \int_0^c x dF(x) = \infty$.

   b). $\sum_{n=1}^\infty P(X > n) \le \sum_{n=1}^\infty \int_{n-1}^n P(X > t)dt = \int_0^\infty P(X > t)dt = E(X)$. On the other hand, $\sum_{n=1}^\infty P(X > n) + 1 \ge \sum_{n=0}^\infty \int_n^{n+1} P(X > t)dt = \int_0^\infty P(X > t)dt = E(X)$. □

4. (Poincaré Formula). If $A_1, ..., A_n$ are events of a probability space $(\Omega, \mathcal{F}, P)$ and

$$T_k = \sum_{1 \le j_1 < j_2 < ... < j_k \le n} P(A_{j_1} A_{j_2} ... A_{j_k}),$$

   then

$$P(\cup_1^n A_j) = \sum_1^n (-1)^{k-1} T_k.$$

   Hint: Use the indicator function.

   Solution.

$$\begin{aligned} & P(\cup_1^n A_j) = 1 - P(\cap_1^n A_j^c) = 1 - E(1_{\cap_1^n A_j^c}) \\ = \quad & 1 - E(\prod_1^n 1_{A_j^c}) = 1 - E(\prod_1^n (1 - 1_{A_j})) \\ = \quad & 1 - \{1 + \sum_{k=1}^n \sum_{1 \le j_1 < j_2 < ... < j_k \le n} (-1)^k E(1_{A_{j_1}} 1_{A_{j_2}} ... 1_{A_{j_k}})\} \end{aligned}$$

$$= \sum_{1}^{n}(-1)^{k-1}T_k.$$

□

5. Show $E(|X+Y|^p) \le 2^p(E(|X|^p)+E(|Y|^p))$, for $p>0$ and any two r.v.s X and Y.

   Solution. $E(|X+Y|^p) \le E[(|X|+|Y|)^p] \le E\{[2(|X| \vee |Y|)]^p\} \le 2^p E\{(|X| \vee |Y|)^p\} \le 2^p E(|X|^p \vee |Y|^p) \le 2^p E(|X|^p + |Y|^p)$. Here $\vee$ is the maximum of two values. □

Exercises

1. Describe the σ-algebra generated from two nonempty sets A and B, where $A \ne B$.

   Solution. Four mutually exclusive sets: $\{AB^c, BA^c, A^cB^c, AB\}$. And the σ-algebra are all the unions of the four sets (totally 11) plus the emtpy set. They are

$$\Big\{ \quad AB^c, BA^c, A^cB^c, AB(\text{the four mutually exclusive sets})$$
$$A, B, (AB)\cup(A^cB^c), (AB^c)\cup(BA^c), B^c, A^c, (\text{union of any two of the four sets})$$
$$B\cup A^c, B^c\cup A, A\cup B, A^c\cup B^c, (\text{union of any three of the four sets})$$
$$\Omega(\text{union of all sets})$$
$$\emptyset.\Big\}$$

   The σ-algebra has totally 16 elements. (Here the product of two sets means intersection). □

2. Given an algebra $\mathcal{F}$, show that the following statements are equivalent:

   (a). $\cup_1^\infty A_n \in \mathcal{F}$, for any $A_n, n \ge 1$ in F.

   (b). $\cap_1^\infty A_n \in \mathcal{F}$, for any $A_n, n \ge 1$ in F.

   (c). $\limsup A_n \in \mathcal{F}$, for any $A_n, n \ge 1$ in F.

   (d). $\liminf A_n \in \mathcal{F}$, for any $A_n, n \ge 1$ in F.

   Solution. Keep in mind that from the definition of algebra that, for any A and B in $\mathcal{F}$, A^c and B^c and $A \cup B$ are in $\mathcal{F}$.

   (a) $\Longrightarrow$ (b): $\cap_n A_n = (\cup_n A_n^c)^c \in \mathcal{F}$.

   (b) $\Longrightarrow$ (a): $\cup A_n = (\cap_n A_n^c)^c \in \mathcal{F}$.

   (b) (and/or) (a) $\Longrightarrow$ (c): $\limsup A_n = \cap_n \cup_{k=n}^\infty A_k \in \mathcal{F}$.

   (c) $\Longrightarrow$ (d): $\liminf_n A_n = \cup_n \cap_{k=n}^\infty A_k = (\cap_n \cup_{k=n}^\infty A_k^c)^c = (\limsup_n A_n^c)^c \in \mathcal{F}$.

   (d) $\Longrightarrow$ (a): $\cup_n A_n = \liminf_n \cup_{k=1}^n A_k \in \mathcal{F}$. □

3. Suppose $\sum_n P(A_n) = \infty$. Show that $\limsup_n P(\cup_{j=1}^n A_j | A_{n+1}) = 1$.

   Solution. Without loss of generality, assume $P(A_n) > 0$ for all $n > 0$. Let $a_n = P(\cap_{j=1}^{n-1} A_j^c A_n)$. Notice that that

$$\sum_{n=2}^\infty a_n = \sum_{n=2}^\infty [P(\cap_{j=1}^{n-1} A_j^c) - P(\cap_{j=1}^{n} A_j^c)] = P(A_1^c) < \infty.$$

   Then,

$$\limsup_n P(\cup_{j=1}^n A_j | A_{n+1}) = \limsup_n \frac{P(\cup_{j=1}^n A_j \cap A_{n+1}}{P(A_{n+1})}$$
$$= \limsup_n \frac{P(A_{n+1}) - P(\cap_{j=1}^n A_j^c A_{n+1})}{P(A_{n+1})} = 1 - \liminf \frac{a_{n+1}}{P(A_{n+1})}$$

Suppose $\liminf a_n/P(A_n) > c > 0$. Then, for all large n, $a_n > cP(A_n)$. This leads to $\sum_n P(A_n) < \infty$, contracting the given condition. Hence $\liminf a_n/P(A_n) = 0$ and $\limsup_n P(\cup_{j=1}^n A_j | A_{n+1}) = 1$. □

4. Let X be a r.v. and g and h are two increasing functions such that $E(g(X)^2) < \infty$ and $E(h(X)^2) < \infty$, show that $corr(g(X), h(X)) \geq 0$. (Hint: Consider the ranges $\{x : g(x) > 0\}$ and $\{x : g(x) < 0\}$).

   Solution. Suppose, for simplicity of argument, $E(g(X)) = 0$. Let a be such that $g(x) \geq 0$ for all $x \geq a$, and $g(x) \leq 0$ for all $x \leq a$. By monotonicity, $h(x) \geq h(a)$ for $x \geq a$ and $h(x) \leq a$ for $x \leq a$. As a result, $g(x)(h(x) - a) \geq 0$ for all x. Therefore

$$E(g(X)h(X)) = E\Big(g(X)[h(X) - h(a)]\Big) \geq 0.$$

   This implies that $\text{corr}(g(X), h(X))$ is ≥ 0.

   If $E(g(X)) \neq 0$, consider $\tilde{g}(x) = g(x) - E(g(X))$. Then, $E(\tilde{g}(X)) = 0$. So $\text{corr}(\tilde{g}(X), h(X))$ is ≥ 0. But $\text{corr}(g(X), h(X))$ is the same as $\text{corr}(\tilde{g}(X), h(X))$.

5. For any $r > 0$, $E(|X|^r) < \infty$ iff $\sum_{n=1}^\infty n^{r-1}P(|X| \geq n) < \infty$.

   Solution. Similar to Problem 3, part a).

$$E(|X|^r) = \int_0^\infty P(|X|^r \geq t)dt = \int_0^\infty P(|X| \geq t^{1/r})dt = \int_0^\infty P(|X| \geq s)ds^r$$
$$= \quad r\int_0^\infty P(|X| \geq s)s^{r-1}ds.$$

   And similar to Problem 3, part b), $E(|X|^r) < \infty$ iff $\sum_{n=1}^\infty n^{r-1}P(|X| \geq n) < \infty$.

6. f is a measurable map from a measurable space $(\Omega, \mathcal{F})$ to another measurable space $(\Omega^*, \mathcal{F}^*)$. Let $\tilde{\Omega} = f(\Omega)$ and $\mathcal{A} = \{A \cap \tilde{\Omega}, A \in \mathcal{F}^*\}$. Show that f is a measurable map from $(\Omega, \mathcal{F})$ to $(\tilde{\Omega}, \mathcal{A})$. (Sorry the original problem is erroneous.)

   Solution. First show $(\tilde{\Omega}, \mathcal{A})$ is a measurable space. For $B \in \mathcal{A}$, $B = A \cap \tilde{\Omega}$ for some $A \in \mathcal{F}^*$. $A^c \cap \tilde{\Omega} = \tilde{\Omega} \setminus A \cap \tilde{\Omega} = \tilde{\Omega} \setminus B$. For $B_j \in \mathcal{A}$, $B_j = A_j \cap \tilde{\Omega}$ for some $A_j \in \mathcal{F}^*$. $\cup_{j=1}^\infty A_j \cap \tilde{\Omega} = \cup_{j=1}^\infty (A_j \cap \tilde{\Omega}) = \cup_{j=1}^\infty B_j$. Hence, $(\tilde{\Omega}, \mathcal{A})$ is a measurable space. For any $B \in \mathcal{A}$, $B = A \cap \tilde{\Omega}$ for some $A \in \mathcal{F}^*$. $f^{-1}(B) = f^{-1}(A \cap \tilde{\Omega}) = f^{-1}(A) \in \mathcal{F}$. It follows that f is a measurable map from $(\Omega, \mathcal{F})$ to $(\tilde{\Omega}, \mathcal{A})$.

7. Suppose $X_1, ..., X_n$ are independent random variables with c.d.f. $F_1, ..., F_n$. Express the c.d.f of $\max\{X_i : 1 \leq i \leq n\}$ and $\min\{X_i : 1 \leq i \leq n\}$ in terms of $F_1, ..., F_n$.

   Solution. $P(\max\{X_i : 1 \leq i \leq n\} \leq t) = P(X_i \leq t, 1 \leq i \leq n) = \prod_i P(X_i \leq t) = \prod_i F_i(t)$.

   $P(\min\{X_i : 1 \leq i \leq n\} \leq t) = 1 - P(\min\{X_i : 1 \leq i \leq n\} > t) = 1 - P(X_i > t, 1 \leq i \leq n) = 1 - \prod_i P(X_i > t) = 1 - \prod_i (1 - F_i(t))$.

8. Suppose $X_n \to 0$ a.e. Show that $P(|X_n| > c, i.o.) = 0$ for all constant $c > 0$.

   Solution. Let $A_n = \{|X_n| > c\}$. Then, for every $\omega \in \{A_n, i.o.\}$, there exists a subsequence $n_k \to \infty$ (depending on ω) such that $|X_{n_k}(\omega)| > c$. Therefore $X_n(\omega) \nrightarrow 0$ for all $\omega \in \{A_n, i.o.\}$. Hence $P(\{A_n, i.o.\}) = 0$.

9. *Solution.* (Method 1). For any $a > 0$, let $X_n^* = \max(-a, \min(X_n, a))$ and $X^* = \max(-a, \min(X, a))$. Then, $E(X_n^*) \to E(X^*)$ as $n \to \infty$. $\sup_n E|X_n^* - X_n| \leq \sup_n E(Y_n 1_{\{Y_n \geq a\}}) \to 0$ as $a \to \infty$. Since $\sup_n E(|X_n^*|) \leq sup_n E(Y_n) < \infty$ for all $a > 0$, we have $\sup_{a>0} E(|X^*|) < \infty$. Hence $E(|X|) < \infty$ (Why?). Then, $E(|X^* - X|) \to 0$ as $a \to \infty$, by dominated convergence theorem. As a result, $E(X_n) \to E(X)$.

   (Method 2). For any $a > 0$, let $X_n^* = \min(X_n^+, a)$ and $X^* = \min(X^+, a)$. Then, $E(X_n^*) \to E(X^*)$ as $n \to \infty$, by the dominated convergence theorem. Observe that $\sup_{n,a} E(X_n^*) \leq sup_n E(Y_n) < \infty$

Hence, $\sup_a E(X^*) < \infty$. Therefore $E(X^+) < \infty$ since $E(X^*) \uparrow E(X^+)$ as $a \uparrow \infty$, by the monotone convergence theorem. Moreover,

$$\sup_n E|X_n^+ - X_n^*| \le \sup_n E(|Y_n - a|1_{\{Y_n > a\}}) \le \sup_n E(Y_n 1_{\{Y_n > a\}}) \to 0$$

as $a \to \infty$. Then,

$$|E(X_n^+) - E(X^+)| \le E|X_n^+ - X_n^*| + |E(X_n^* - E(X^*)| + |E(X^*) - E(X^+)| \to 0$$

as $n \to \infty$ and then $a \to \infty$. Similarly, one can show $E(X_n^-) \to E(X^-)$. □

10. For events $A_1, ..., A_n$, set $q_k = \sum_{j_1 < ... < j_k} P(A_{j_1} \cdots A_{j_k})$. For every even positive $m \le n$, show that

$$\sum_{j=1}^{m} (-1)^{k-1} q_k \le P(\cup_{j=1}^n A_j) \le \sum_{j=1}^{m-1} (-1)^{k-1} q_k.$$

Solution. For any $1 \le m \le J$,

$$1 - \sum_{j=1}^{m} (-1)^{j-1} \binom{J}{j} \qquad \begin{cases} \le 0 & \text{for odd } m; \\ \ge 0 & \text{for even } m. \end{cases}$$

(DIY: Please verify using the fact that $\sum_{j=0}^{J} (-1)^j \binom{J}{j} = 0$, $\binom{J}{j} = \binom{J}{J-j}$ and $\binom{J}{j}$ is increasing for $j \le J/2$.) Write $\cup_{j=1}^n A_j = \cup_{k=1}^n B_k$ where B_k is the set of ω which belongs to exactly k of the sets $A_1, ..., A_n$. For $\omega \in B_J$,

$$\sum_{k=1}^{m} (-1)^{k-1} \sum_{j_1 < ... < j_k} 1_{A_1}(\omega) \cdots 1_{A_k}(\omega) = \begin{cases} \sum_{k=1}^{m} (-1)^{k-1} \binom{J}{k} & \text{for } m < J \\ \sum_{k=1}^{J} (-1)^{k-1} \binom{J}{k} & \text{for } m \ge J \end{cases}$$
$$= \begin{cases} \sum_{k=1}^{m} (-1)^{k-1} \binom{J}{k} & \text{for } m < J \\ 1 & \text{for } m \ge J \end{cases}$$
$$\begin{cases} \le 1 & \text{if } m \text{ is even} \\ \ge 1 & \text{if } m \text{ is odd} \end{cases}$$

Let J be $1, 2, ..., n$. Then, the above inequality implies for $\omega \in \cup_{J=1}^n B_J = \cup_{j=1}^n A_j$

$$\sum_{k=1}^{m} (-1)^{k-1} \sum_{j_1 < ... < j_k} 1_{A_1}(\omega) \cdots 1_{A_k}(\omega) \begin{cases} \le 1 & \text{if } m \text{ is even} \\ \ge 1 & \text{if } m \text{ is odd} \end{cases}$$

As the left hand side is 0 if $\omega \notin \cup_{J=1}^n B_J = \cup_{j=1}^n A_j$. We therefore have

$$sum_{k=1}^{m} (-1)^{k-1} \sum_{j_1 < ... < j_k} 1_{A_1}(\omega) \cdots 1_{A_k}(\omega) \begin{cases} \le 1_{\cup_{j=1}^n A_j} & \text{if } m \text{ is even} \\ \ge 1_{\cup_{j=1}^n A_j} & \text{if } m \text{ is odd.} \end{cases}$$

Taking expectation on both sides, the Bonferroni's inequality follows. □