Solutions of Homework 2.

1. Suppose $|X_n| \le Y$ with $E(Y) < \infty$, and $X_n \to X$ in probability. Show that $E|X_n - X| \to 0$ (i.e., $X_n \to X$ in L_1).

   Solution. Method 1. Suppose $X_n \not\to X$ in L_1. Then, there exists a subsequence $\{n_k\}$ such that $E|X_{n_k} - X| > \epsilon$ for some $\epsilon > 0$ and all $\{n_k\}$. Since $X_{n_k} \to X$ in probability, there exists a further subsequence n_{k_j} such that $X_{n_{k_j}} \to X$ a.e.. Then the dominated convergence theorem implies that $E|X_{n_{k_j}} - X| \to 0$. This contradicts with $E|X_{n_k} - X| > \epsilon$ for some $\epsilon > 0$ and all $\{n_k\}$. Hence, $X_n \to X$ in L_1.

   Method 2. Let $Z_n = |X_n - X|$. Then $Z_n \le 2Y$.

$$
\begin{aligned}
E(Z_n) &= E(Z_n 1_{\{Z_n \le \epsilon\}}) + E(Z_n 1_{\{Z_n > \epsilon\}}) \\
&\le \epsilon + 2E(Y 1_{\{Z_n > \epsilon\}}) \\
&\le \epsilon + 2E(Y 1_{\{Y > 1/\epsilon, Z_n > \epsilon\}}) + 2E(Y 1_{\{Y \le 1/\epsilon, Z_n > \epsilon\}}) \\
&\le \epsilon + 2E(Y 1_{\{Y > 1/\epsilon\}}) + 2/\epsilon E(1_{\{Z_n > \epsilon\}}) \\
&= \epsilon + 2E(Y 1_{\{Y > 1/\epsilon\}}) + 2/\epsilon P(Z_n > \epsilon) \\
&\to \epsilon + E(Y 1_{\{Y > 1/\epsilon\}}) \qquad \text{as } n \to \infty
\end{aligned}
$$

   which can be arbitrarily small since ϵ can be chosen arbitrarily small. Therefore $\lim_n E(Z_n) = 0$. □

2. $X_n \to X$ in probability if and only if, for any sub-sequence of $X_n, n \ge 1$, there always exists a further sub-sequence, which may be called sub-sub-sequence, that converges a.s. to X.

   Solution. " $\Longleftarrow$" Suppose X_n does not converge to X in probability. There exists a subsequence n_k and an $\epsilon > 0$ such that $P(X_{n_k} - X| > \epsilon) > \epsilon$. Then, no further subsequence of X_{n_k} can converge to X, a.s. or in probability, which leads to a contradiction.

   "$\Longrightarrow$" Any subsequence of $\{X_n\}$ still converges to X in probability. And there is always of further subsequence of this subsequence that converges to X a.s.. □

3. Two r.v.s X, Y are called conditionally independent give r.v. Z if $P(X \le t, Y \le s|Z = z) = P(X \le t|Z = z)P(Y \le s|Z = z)$ for all t, s, z. Let X_n be the total number of heads of the first n tosses of a fair coin. Set $X_0 = 0$. Show that X_{n-1} and X_{n+1} are conditionally independent given X_n.

   Solution. Let ξ_n be 1 or 0 when the n-th toss is a head or tail. Set $\xi_0 = 0$. Then $\xi_i, i = 0, 1, 2, ...$ are independent. $X_n = \sum_{j=0}^n \xi_j$. And ξ_{n+1} is independent of X_n and ξ_n

$$
\begin{aligned}
& P(X_{n+1} = k, X_{n-1} = j|X_n = l) \\
=\ & P(\xi_{n+1} = k - l,\ \xi_n = l - j|X_n = l) \\
=\ & \frac{P(\xi_{n+1} = k - l,\ \xi_n = l - j, X_n = l)}{P(X_n = l)} \\
=\ & P(\xi_{n+1} = k - l)\frac{P(\xi_n = l - j,\ X_n = l)}{P(X_n = l)} \\
=\ & P(\xi_{n+1} = k - l|X_n = l)P(\xi_n = l - j|X_n = l) \\
=\ & P(X_n = k|X_n = l)P(X_{n-1} = j|X_n = l)
\end{aligned}
$$

   □

4. Suppose $X_1, X_2, ...$ are i.i.d. random variables following exponential distribution with mean 1. Show that $Y_n \to \infty$ a.e., where $Y_n = \max_{1 \le i \le n} X_i$. (You might consider first show the convergence in probability.)

Solution. For any constant $C > 0$,

$$P(Y_n > C) = P(\max_{1\le i\le n} X_i > C) = 1 - P(\max_{1\le i\le n} X_i \le C) = 1 - (1-e^{-C})^n \to 1$$

as $n \to \infty$. Therefore $Y_n \to \infty$ in probability. Since Y_n is nondecreasing, $Y_n \to \infty$ a.e.. (why?) □

5. Raise an example to show that Fatou's lemma does not hold if the condition of $X_n \ge 0$ is dropped.

   Solution. Suppose $\xi \sim Unif[0,1]$. Let $X_n = -1/\xi 1_{\{\xi \le 1/n\}}$. Then $X_n \to 0$ a.e., but $E(X_n) = -\infty$ for all n. Therefore $E(\liminf X_n) = 0 > \liminf_n E(X_n) = -\infty$. □

DIY Exercises

1. ⋆⋆ (Extension of Fatou's lemma). Suppose $X_n \ge Y$ and $E(Y^-) < \infty$. Then $E(\liminf_n X_n) \le \liminf_n E(X_n)$.

   Solution. $X_n \ge Y \Longrightarrow X_n - Y \ge 0 \Longrightarrow X_n - (Y^+ - Y^-) \ge 0 \Longrightarrow X_n + Y^- \ge 0$. By Fatou's lemma,

$$\begin{aligned} E(\liminf_n X_n) &= E(\liminf_n (X_n + Y^-) - E(Y^-) \\ \le\ & \liminf_n E(X_n + Y^-) - E(Y^-) = \liminf_n E(X_n). \end{aligned}$$

□

2. ⋆⋆⋆⋆ Suppose $|X_n| \le Y_n$, $E(|Y_n - Y|) \to 0$ with $E(Y) < \infty$, and $X_n \to X$ in probability. Then $E(X_n) \to E(X)$.

   Solution. (This problem is a further extension of Problem 1)

   Method 1: $|X_n - X| \le Y_n + Y$. Let $\xi_n = |X_n - X|$. Then $\xi_n \to 0$ in probability. Choose any $0 < \epsilon < c < \infty$. Then,

$$\begin{aligned} E(\xi_n) &= E(\xi_n 1_{\{\xi_n \le c\}}) + E(\xi_n 1_{\{\xi_n > c\}}) \\ &\le E(\xi_n 1_{\{\xi_n \le c\}}) + E(\xi_n 1_{\{\xi_n > c\}}) + E(\xi_n 1_{\{\xi_n > c\}}) \\ &\le \epsilon P(\xi_n \le \epsilon) + E(\xi_n 1_{\{\epsilon < \xi_n \le c\}}) + E[(Y_n + Y)1_{\{Y_n + Y > c\}} \\ &\le \epsilon + cP(\xi_n > \epsilon) + E(|Y_n - Y|) + 2E(Y 1_{\{Y_n + Y > c\}}) \\ &\to \epsilon + 0 + 0 + E(Y 1_{\{2Y > c\}}). \end{aligned}$$

   Letting $\epsilon \downarrow 0$ and $c \uparrow \infty$, we have $E(\xi_n) \to 0$.

   Method 2: Assume first $X_n \to X$ a.e. and $Y_n \to Y$ a.e.. Then Fatou's lemma implies

$$E(Y - X) = E(\liminf_n (Y_n - X_n)] \le \liminf E(Y_n - X_n) = E(Y) - \limsup_n E(X_n)$$

$$E(Y + X) = E(\liminf_n (Y_n + X_n)] \le \liminf E(Y_n + X_n) = E(Y) + \liminf_n E(X_n)$$

   So, the limit of $E(X_n)$ exists and equals to $E(X)$. Now suppose $E(X_n) \nrightarrow E(X)$. There exists a subsequence $\{n_k\}$ such that $|E(X_{n_k} - E(X)| > \epsilon$ for some $\epsilon > 0$ and all $\{n_k\}$. But for this subsequence, since $X_{n_k} \to X$ and $Y_n \to Y$ in probability, there exists a further subsequence $\{X_{n_{k_j}}\} \to X$ a.e. and $\{Y_{n_{k_j}}\} \to Y$ a.e.. Then the above proof implies $E(X_{n_{k_j}}) \to E(X)$, which contracts with $|E(X_{n_k}) - E(X)| > \epsilon$. □

3. ⋆ Show that $Bin(n, p_n) \to \mathcal{P}(\lambda)$ if $n \to \infty$ and $np_n \to \lambda > 0$. (This problem and the next one are for your knowledge about the basic facts of commonly used distributions.)

Solution. For any fixed integer $k \geq 0$,

$$P(Bin(n,p_n)=k) = \binom{n}{k} p_n^k (1-p_n)^{n-k} = \frac{n!}{k!(n-k)!}\Big(\frac{p_n}{1-p_n}\Big)^k (1-p_n)^n$$
$$= \frac{n(n-1)\cdots(n-k+1)}{k!n^k}\Big(\frac{np_n}{1-p_n}\Big)^k (1-p_n)^n \to \frac{1}{k!}\lambda^k e^{-\lambda}.$$

□

4. ⋆ Suppose M and N are two independent Poisson random variables with mean λ and θ. Show that $M+N$ is still Poisson random variable, and, moreover, the conditional distribution of M given $M+N=k$ is $Bin(k,p)$, where $p=\lambda/(\lambda+\theta)$.

   Solution.

$$P(M+N=k) = \sum_{j=0}^{k} P(M=j, N=k-j) = \sum_{j=0}^{k} P(M=j)P(N=k-j)$$
$$= \sum_{j=0}^{k} \frac{1}{j!}\lambda^j e^{-\lambda}\frac{1}{(k-j)!}\theta^{k-j}e^{-\theta} = \frac{1}{k!}(\lambda+\theta)^k e^{-\lambda-\theta}\sum_{j=0}^{k}\binom{k}{j}\Big(\frac{\lambda}{\lambda+\theta}\Big)^j\Big(\frac{\theta}{\lambda+\theta}\Big)^{k-j}$$
$$= \frac{1}{k!}(\lambda+\theta)^k e^{-\lambda-\theta}.$$

   And

$$P(M=j|M+N=k) = \frac{P(M=j,N=k-j)}{P(M+N=k)} = \frac{P(M=j)P(N=k-j)}{P(M+N=k)}$$
$$= \frac{k!}{j!(k-j)!}\frac{\lambda^j e^{-\lambda}\theta^{k-j}e^{-\theta}}{(\lambda+\theta)^k e^{-\lambda-\theta}} = \binom{k}{j}p^j(1-p)^{k-j}$$

   where $p=\lambda/(\lambda+\theta)$. □

5. ⋆ Suppose $X \in \mathcal{F}$ (meaning that X is measurable to a σ-algebra $\mathcal{F}$). Show that $E(X|\mathcal{F}) = X$, a.e.

   Solution. Since X satisfies condition $X \in \mathcal{F}$ and the condition $E(X1_A) = E(X1_A)$ for every $A \in \mathcal{F}$, which is actually an identity, so $E(X|\mathcal{F}) = X$, a.s.. □

6. ⋆⋆ Suppose $X_n \geq 0$ and $X_n \uparrow X$ a.e., then $E(X_n|\mathcal{F}) \uparrow E(X|\mathcal{F})$, a.e.. (This is the monotone convergence theorem for conditional expectation. Fatou's lemma and the dominated convergence theorem also hold for conditional expectation.)

   Solution. Since $X_n \uparrow$, $X_{n+1} - X_n \geq 0$. Therefore $E(X_{n+1} - X_n|\mathcal{F}) \geq 0$. Hence, $E(X_n|\mathcal{F}) \uparrow$. Let the limit be ξ. Since $E(X_n|\mathcal{F}) \in \mathcal{F}$, the limit ξ is also $\mathcal{F}$-measurable. For any set $A \in \mathcal{F}$, $E(X_n|\mathcal{F})1_A \uparrow \xi 1_A$. by monotone convergence theorem,

$$E(\xi 1_A) = \lim_n E[E(X_n|\mathcal{F})1_A] \qquad \text{by monotone convergence theorem}$$
$$= \lim_n E(X_n 1_A) \qquad \text{by the definition of conditional expectation w.r.t. } \sigma\text{-algebra}$$
$$= E(X1_A) \qquad \text{by monotone convergence theorem.}$$

   Therefore by the definition of conditional expectation with respect to a σ-algebra, $\xi = E(X|\mathcal{F})$. □

7. ⋆ Suppose $X_n \to c$ in distribution where c is a constant. Then $X_n \to c$ in probability.

   Solution. For constant c as a r.v., its c.d.f. $F(t) = 1$ for all $t \geq c$ and $F(t) = 0$ for all $t < c$. Therefore $P(X_n \leq t) \to 1$ for any $t > c$ and $P(X_n \leq t) \to 0$ for any $t < c$. Hence, $P(X_n > t) \to 0$ for any $t > c$. So $X_n \to c$ in probability. □

8. ★★ Let $X_1, X_2, ...$ be i.i.d. r.v.s. with $\limsup_{t\to\infty} tP(X_1 > t) \to 0$. Show that $Y_n/n \to 0$ in probability, where $Y_n = \max_{1\le i\le n} X_i$. (Hint: use Chebyshev inequality).

   Solution. Let $\epsilon > 0$.

$$\begin{aligned} & P(Y_n/n > \epsilon) = P(\max_{1\le i\le n} X_i > n\epsilon) = 1 - P(\max_{1\le i\le n} X_i \le n\epsilon) \\ = \;& 1 - P(X_1 \le n\epsilon)^n = 1 - (1 - P(X_1 > n\epsilon))^n \\ = \;& 1 - e^{n\log(1-P(X_1>n\epsilon))} \approx 1 - e^{-nP(X_1>n\epsilon)} \to 0 \end{aligned}$$

   Next,

$$P(Y_n/n < -\epsilon) = P(\max_{1\le i\le n} X_i < -n\epsilon) \le P(X_1 < -n\epsilon) \to 0.$$

   So, $Y_n/n \to 0$ in probability. □

9. ★ Let f_n and f be bounded continuous functions such that $\lim_n \sup_t |f_n(t) - f(t)| = 0$. Suppose $X_n \to X$ in distribution. Then, $E(f_n(X_n)) \to E(f(X))$.

   Solution.

$$\begin{aligned} & |E(f_n(X_n)) - E(f(X))| \le E|f_n(X_n) - f(X_n)| + |E(f(X_n)) - E(f(X))| \\ \le \;& \sup_t |f_n(t) - f(t)| + |E(f(X_n)) - E(f(X))| \to 0. \end{aligned}$$

   □

10. ★ For any sequence of r.v.s. X_n, there exists a sequence of constants A_n such that $X_n/A_n \to 0$ a.e.. (Hint: use Borel-Contelli Lemma).

   Solution. Choose a_n such that $P(|X_n| > a_n) \le 1/2^n$. Let $A_n = na_n$. Then,

$$|X_n| 1_{\{|X_n|\le a_n\}}/A_n \le a_n/A_n = 1/n \to 0.$$

   And $P(|X_n| > a_n, i.o.) = 0$ as $\sum_{n=1}^{\infty} P(|X_n| > a_n) < \infty$ by the Borel-Cantelli lemma. Therefore,

$$|X_n| 1_{\{|X_n|>a_n\}}/A_n \to 0, \qquad \text{a.e.}$$

   Consequently, $X_n/A_n \to 0$ a.e.. □

11. ★ Suppose $\mathcal{F} \subseteq \mathcal{A}$. Show that $E(E(X|\mathcal{F})|\mathcal{A}) = E(X|\mathcal{F})$.

   Solution. Let $Y = E(X|\mathcal{F})$. Y is $\mathcal{F}$-measurable, so Y must be $\mathcal{A}$-measurable since $\mathcal{F} \in \mathcal{A}$. Therefore $E(Y|\mathcal{A} = Y$. □

12. ★★ (CRUDE VERSION OF MARTINGALE CONVERGENCE THEOREM) Suppose $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ for $n \ge 1$. Let $\mathcal{F} = \sigma(\cup_{n=1}^{\infty} F_n)$. For any random variable X with $|X| \le c > 0, a.s.$, assume $E(X|\mathcal{F}_n)$ has an a.s. limit. show that

$$E(X|\mathcal{F}_n) \to E(X|\mathcal{F}), \qquad a.s.$$

   Solution. Perhaps for a better understanding, denote $Y_n = E(X + c|\mathcal{F}_n)$ and $Y = E(X + c|\mathcal{F})$, which are nonnegative r.v.s. For any $A \in \mathcal{F}_m$, $E(Y_n 1_A) = E(Y 1_A) = E(X^+ 1_A)$ for all $n \ge m$. Then, Fatou's lemma ensures

$$E(\liminf_n Y_n 1_A) \le \liminf_n E(Y_n 1_A) = E(Y 1_A).$$

   i.e.,

$$E((Y - \liminf_n Y_n) 1_A) \ge 0, \qquad \text{for all } A \in \mathcal{F}_m, m \ge 1$$

   It implies $E((Y - \liminf_n Y_n)1_A) \ge 0$, for all $A \in \mathcal{F}$, which can be proved by showing $\{A \in \mathcal{F} : E((Y - \liminf_n Y_n)1_A) \ge 0\}$ is a σ-algebra which contains $\mathcal{F}_m$, $m \ge 1$, and therefore must be the same as $\mathcal{F}$. Then, $Y - \liminf_n Y_n$, being $\mathcal{F}$-measurable, must be nonnegative a.s.. As a result, we have shown

$$\liminf_n E(X|\mathcal{F}_n) \leq E(X|\mathcal{F}), \qquad a.s.$$

Next, by considering $Y_n = E(c - X|\mathcal{F}_n)$ and $Y = E(c - X|\mathcal{F})$, one can likewise show

$$\limsup_n E(X|\mathcal{F}_n) \geq E(X|\mathcal{F}). \qquad a.s.$$

Consequently, $\lim_n E(X|\mathcal{F}_n) = E(X|\mathcal{F})$ $a.s.$. □