**Solution to DIY Problems in Homework 3.**

1. ⋆⋆ Suppose $X_1, X_2, \ldots$ are independent nonnegative r.v.s.. Then, $\sum_n X_n < \infty$ a.s. if and only if $\sum_n P(X_n > c) < \infty$ and $\sum_n E(X_n \wedge c) > \infty$ for all or any $c > 0$.

   *Solution.* As $\sum_n X_n < \infty$ is equivalent to By $\sum_n X_n/c < \infty$ for all or any $c > 0$. We only need to consider $c = 1$ and cite Kolmogorov's three series theorem. For the sufficiency, the convergence of the first two series is trivial as the third also holds since $var(X_n \wedge 1) \leq E((X_n \wedge 1)^2) \leq E(X_n \wedge 1)$. For the necessary, the convergence of the first series is $\sum_n P(X_n > 1) < \infty$ and the $\sum_n E(X_n \wedge 1) = \sum_n E(X_n 1_{\{X_n \leq 1\}}) + \sum_n P(X_n > 1) < \infty$ by the convergence of the first and second series. □

2. ⋆⋆ Suppose $X_1, X_2, \ldots$ are independent nonnegative r.v.s. Prove that $\sum_n X_n < \infty$ a.s. if and only if $\sum_n E[X_n/(1 + X_n)] < \infty$.

   *Solution.* Notice that $E[X_n/(1 + X_n)1_{\{X_n>1\}}] \leq P(X_n > 1) \leq 2E[X_n/(1 + X_n)1_{\{X_n>1\}}]$ and $E(X_n/(1+X_n)1_{\{X_n\leq 1\}}) > E(X_n 1_{X_n \leq 1}) > E(X_n/(1+X_n)1_{\{X_n\leq 1\}})$. The the desired claim follows from Kolmogorov's three series theorem and that the $X_i$ are nonnegative. □

3. Let $\xi_1, \xi_2, \ldots$ be iid Cauchy r.v.s. with common density $1/[\pi(1+x^2)]$ and let $X_n = |\xi_n|$. Find $b_n \uparrow \infty$ such that $S_n/b_n \to 1$ in probability.

   *Solution.* Notice that

$$E(|\xi|1_{\{|\xi|<a\}}) = \int_0^a \frac{2t}{\pi(1+t^2)}dt = \frac{1}{\pi}\log(1+a^2) \approx \frac{2}{\pi}\log a, \qquad \text{for large } a.$$

   and

$$P(|\xi| > a) = \int_a^\infty \frac{2}{\pi(1+t^2)}dt \approx \frac{2}{\pi a}, \qquad \text{for large } a.$$

   We check the two conditions of the WLLN. Let $b_n = (2/\pi)n\log(n)$ and $a_n = n\log n$. Then $nP(|\xi| > a_n) = 1/\log n \to 0$. and

$$nE(|\xi|^2 1_{|\xi|\leq a_n})/b_n^2 \leq \frac{1}{\pi}\frac{na_n}{b_n^2} = \frac{n^2\log n}{\pi(n\log n)^2} \to 0.$$

   Then, $S_n/b_n \to 1$ in probability. (In fact, no $b_n \uparrow \infty$ exists such that $S_n/b_n \to 1$ a.s..) □

4. Suppose $X_1, \ldots, X_n, \ldots$ are i.i.d. with mean 1. Show that $\max_{1\leq k\leq n}|X_k|/|S_n| \to 0$ a.s.

   *Solution.* By the SLLN, $S_n/n \to 1$ a.e. It suffices to show $\max_{1\leq k\leq n}|X_k|/n \to 0$ a.e.. First, since $X_i$ has finite mean, we know $E|X| < \infty$ and therefore $\sum_{i=1}^\infty P(|X_i| > i/\epsilon) < \infty$. It follows from the Borel-Cantelli lemma that, with probability 1, $X_n/n \leq \epsilon$ for all large $n$. In other words, $X_n/n \to 0$ a.e.. Write

$$\begin{aligned}
&\max_{1\leq k\leq n}|X_k|/n \leq \max_{1\leq k\leq M}|X_k|/n + \max_{M\leq k\leq n}|X_k|/n \\
\leq\ & \max_{1\leq k\leq M}|X_k|/n \max_{M\leq k\leq n}|X_k|/k \leq \max_{1\leq k\leq M}|X_k|/n + \max_{M\leq k\leq \infty}|X_k|/k \\
\to\ & \max_{M\leq k\leq\infty}|X_k|/k \quad \text{a.e.,} \quad \text{by letting } n\to\infty \\
\to\ & 0, \quad \text{a.e.,} \quad \text{by letting } M\to\infty.
\end{aligned}$$

□

5. Suppose $X_1, ...X_n, ...$ are independent with mean 0 and variance 1. Suppose $c_n$ are constants such that $c_n/n^p \downarrow 0$ for some $0 < p < 1/2$. Show that $\sum_{j=1}^n c_j X_j/n \to 0$ a.s.

   *Solution.*
$$\sum_{n=1}^{\infty} \text{var}(c_n X_n)/n^2 = \sum_{n=1}^{\infty} c_n^2/n^2 \le \sum_{n=1}^{\infty} \left(\frac{c_n}{n^p}\right)^2 \frac{1}{n^{2-2p}} < \infty.$$

   By Kolmogorov's criterion of SLLN, $\sum_{j=1}^n c_j X_j/n \to 0$, a.e.. □

6. Prove, for iid r.v.s $X, X_1, X_2, ...$, that $(S_n - C_n)/n \to 0$ a.s. for some constants $C_n$ if and only if $E(|X|) < \infty$. Hint: Symmetrization.

   *Solution.* "$\Longleftarrow$" If $E|X| < \infty$, choose $C_n = nE(X)$. Then, The SLLN ensures $(S_n - C_n)/n \to 0$ a.s..

   "$\Longrightarrow$" Let $X_i^*, i = 1, 2, ..$ be iid independent of $X_n, n = 1, 2, ....$ Then $(\sum_{i=1}^n X_i^* - C_n)/n \to 0$ a.s.. As a result, $\sum_{i=1}^n (X_i - X_i^*)/n \to 0$, a.s.. Kolmogorov's SLLN implies $E(|X_1 - X_1^*|) < \infty$. Therefore $E(|X|) < \infty$. □

7. If $X, X_1, X_2, ...$ are iid with $E(|X|^p) = \infty$ for some $p > 0$, then $\limsup |S_n|/n^{1/p} = \infty$ a.s.

   *Solution.* $E(|X|^p) = \infty$ ensures $\limsup |X_n|/n^{1/p} \to \infty$ a.s., which also ensures $\limsup |S_n|/n^{1/p} = \infty$ a.s.. □

8. Suppose $X, X_1, X_2, ...$ are iid with $E(|X|^p) < \infty$ for some $p \ge 1$. Then, $E(|S_n/n|^p) \to |E(X)|^p$. Hint: Fatou lemma.

   *Solution.* It follows from SLLN that $S_n/n \to E(X)$ a.s. and $\sum_{i=1}^n |X_i|^p/n \to E(|X|^p)$ a.s.. Then,

$$\begin{aligned} \liminf[E|X|^p - E(|S_n/n|^p)] &= \liminf[E(\sum_{i=1}^n E|X_i|^p/n) - E(|S_n/n|^p)] \\ &\ge E[\liminf(sum_{i=1}^n |X_i|^p/n)] - E(\limsup |S_n/n|^p) = E(|X|^p) - |E(X)|^p. \end{aligned}$$

   As a result, $\limsup E(|S_n/n|^p) \le |E(X)|^p$. On the other hand, $\liminf E(|S_n/n|^p) \ge E(|\liminf S_n/n|^p) = |E(X)|^p$. Then, $E(|S_n/n|^p) \to |E(X)|^p$. □

9. Suppose $X, X_1, X_2, ...$ are iid with mean 1 and $a_n$ are bounded real numbers. Then, $(1/n)\sum_{j=1}^n a_j \to 1$ if and only if $(1/n)\sum_{j=1}^n a_j X_j \to 1$ a.s.. Hint: Repeat the proof of Kolmogorov's SLLN.

   *Solution.* Let $\xi = X - 1$ and $\xi_i = X_i - 1$. Then, $\xi, \xi_1, \xi_2, ...$ are iid with mean 0. We only need to show $(1/n)\sum_{j=1}^n a_j \xi_j \to 0 a.s.$. Let $\eta_i = \xi_i 1_{\{|\xi_i \ge i\}}$.

   (i) $P(\xi_i \ne \eta_i,\ i.o.) = 0$. Thus $(1/n)\sum_{i=1}^n a_i \xi_i \to 0$ a.s. is equivalent to Thus $(1/n)\sum_{i=1}^n a_i \eta_i \to 0$ a.s..

   (ii) $E(\eta_n) \to 0$. Therefore $a_n \eta_n \to 0$ as $a_n$ are bounded. Hence $(1/n)\sum_{i=1}^n a_i E(\eta_i) \to 0$.

   (iii)
$$\sum_{i=1}^{\infty} \text{var}(a_i \eta_i/i) \le \sup_k a_k^2 \sum_{i=1}^{\infty} \text{var}(\eta_i)/i^2 < \infty$$

   Therefore $\sum_{i=1}^{\infty} a_i(\eta_i - E(\eta_i))/i < \infty$ a.s.. And the Kronecker lemma implies $(1/n)\sum_{i=1}^n a_i(\eta_i - E(\eta_i)) \to 0$ a.s..

   Combine (i), (ii) and (iii), we have $(1/n)\sum_{j=1}^n a_j \xi_j \to 0 a.s.$. □

10. Suppose $X_1, ..., X_n, ...$ are iid with common density $f(x) = cg(x)$ with $g(x) = x^{-\alpha}$ if $x > 1$, 1 if $|x| \le 1$ and $(-x)^{-\beta}$ if $x < -1$. $c > 0$ is some constant and $\alpha > 1$ and $\beta > 2$. Find the a.e. limit of $S_n/n$ in terms of $\alpha$ and $\beta$. What condition on $\alpha$ and $\beta$ would ensure $\sum_{n=1}^{\infty} X_n/n < \infty$ a.e.? (Hint: verify the three conditions in Kolmogorov's three series theorem; first make sure that $\alpha = \beta$.)

*Solution.*

$$E(X_n/n1_{\{|X_n|/n\le 1\}}) = (c/n)[\int_1^n xx^{-\alpha}dx + \int_{-n}^{-1} x(-x)^{-\beta}dx + \int_{-1}^{1} xdx]$$
$$= (c/n)\Big(\frac{n^{-\alpha+2}-1}{-\alpha+2} - \frac{n^{-\beta+2}-1}{-\beta+2}\Big)$$

(If $\alpha = 2$, $\frac{n^{-\alpha+2}-1}{-\alpha+2}$ is replaced by $\log n$.) $\sum_{n=1}^{\infty} E(X_n/n1_{\{|X_n|/n\le 1\}}) < \infty$ only if $\alpha = \beta$. Condition (ii) holds only when $\alpha = \beta$.

For condition (i), now that $\alpha = \beta$,

$$\sum_n P(|X_n|/n > 1) = \sum_n P(|X_1| > n) = 2c\sum_n \int_n^{\infty} x^{-\beta}dx = 2c\sum_n n^{-\beta+1}/(\beta-1) < \infty,$$

since $\beta > 2$.

For condition (iii),

$$\sum_n var(X_n/n1_{\{|X_n|\le n\}}) = \sum_n (2c/n^2)[\int_1^n x^2x^{-\beta}dx + \int_0^1 x^2dx]$$
$$= \sum_n (2c/n^2)\Big(\frac{n^{-\beta+3}-1}{-\beta+3} + 1/3\Big) < \infty,$$

since $\beta > 2$. (Here $\frac{n^{-\beta+3}-1}{-\beta+3}$ is replaced by $\log(n)$ if $\beta = 3$.

Overall, the condition on $\alpha$ and $\beta$ is $\alpha = \beta$ to ensure $\sum_n X_n/n < \infty$, a.s.. □

11. For any sequence of $\sigma_n^2$ satisfying $\sum_n \sigma_n^2/n^2 = \infty$, raise an example of a sequence of independent random variables $X_1, X_2, ...$ with mean 0 and $\text{var}(X_n) \le \sigma_n^2$ such that $X_n/n$ does not converge to 0 a.s., and, as a result, $\sum_n X_n/n \not< \infty$ a.s..

*Solution.* Without loss of generality assume $0 < \sigma_n \le n$. Let $a_n = \sigma_n/n$ and $X_n = \pm n$ with probability $a_n^2/2$ and 0 with probability $1 - a_n$. Then $X_n$ are independent with mean 0 and variance $\sigma_n^2$. And, for any $0 < \epsilon < 1$, $P(|X_n|/n > \epsilon) = a_n^2$ and therefore $\sum_n P(|X_n|/n > \epsilon) = \infty$. It follows from Borel-Cantelli lemma that $P(\{X_n/n \to 0\}) = 0$. □

12. Suppose $\xi_1, ..., \xi_n$ is a sequence of r.v.s such that, for $p > 0$,

$$\sup_{i,j\ge n} E(|\xi_i - \xi_j|^p) \to 0, \qquad \text{as} \qquad n \to \infty.$$

Then, there exists a r.v., denoted as $\xi$, such that $\xi_n \to \xi$ in $L^p$.

*Solution.* There exists $n_k \uparrow \infty$ such that $\sup_{i,j\ge n_k} E(|\xi_i - \xi_j|^p) \le 2^{-2pk}$. Let $A_k = \{|\xi_{n_k} - \xi_{n_{k+1}}| \ge 2^{-k}\}$. By Markov inequality,

$$P(A_k) \le \sup_{i,j\ge n_k} P(|\xi_i - \xi_j| \ge 2^{-k}) \le \sup_{i,j\ge n_k} E(|\xi_i - \xi_j|^p)/2^{-pk} \le 2^{-pk}.$$

Borel-Cantelli lemma implies $P(A_k \ i.o.) = 0$. Hence, $P(\liminf_k A_k^c) = 1$. But on $\liminf_k A_k^c$, $\xi_{n_k}$ is a Cauchy sequence. By the Cauchy criterion, there exists a limit, denoted as $\xi$, of $\xi_{n_k}$ a.s.. Then,

$$\lim_n E(|\xi_n - \xi|^p) = \lim_n E(|\xi_n - \lim_k \xi_{n_k}|^p)$$
$$\le \lim_n \liminf_k E(|\xi_n - \xi_{n_k}|^p) \qquad \text{by Fatou's Lemma}$$
$$\le \lim_n \sup_{j\ge n} E(|\xi_j - \xi_n|^p) \to 0.$$

□

13. ⋆⋆⋆⋆ Suppose $X_n$ are iid Poisson random variable with rate $\lambda > 0$. Prove that

$$\limsup_n \frac{X_n \log\log n}{\log n} = 1 \qquad a.s..$$

Hint: first show $P(X = n) \le P(X \ge n) \le e^{\lambda} P(X = n)$.

*Solution.* Write

$$P(X \ge n) = \sum_{k=n}^{\infty} e^{-\lambda}\lambda^k/k! = \frac{e^{-\lambda}\lambda^n}{n!}\sum_{k=n}^{\infty}\frac{\lambda^{k-n}}{k!/n!} \le \frac{e^{-\lambda}\lambda^n}{n!}\sum_{k=n}^{\infty}\frac{\lambda^{k-n}}{(k-n)!} = P(X=n)e^{\lambda}.$$

Let $a_n$ be the integer part of $\delta \log n/\log\log n$ for a $\delta > 0$. Then,

$$P(X = a_n) = \frac{e^{-\lambda}\lambda^{a_n}}{a_n!} = e^{-\lambda}e^{a_n \log\lambda}e^{-\sum_{j=1}^{a_n}\log j} = e^{-a_n \log a_n(1+o(1))} = n^{-\delta+o(1)}.$$

Then, $\sum_{n=1}^{\infty} P(X_n \ge a_n) < \infty$ or $= \infty$ depending upon $\delta > 1$ or $\delta < 1$. This implies, by the Borel-Cantelli lemma,

$$\limsup \frac{X_n}{\log n/\log\log n} = 1 \qquad a.s..$$

□