PRELIMINARIES

Some Useful Notations

$\exists$	there exists
$\forall$	for all
$\Rightarrow$	implies
$\Leftrightarrow$	equivalent to (iff)
$\in$	belongs to
$\{x \mid q(x)\}$	set specification
$A \subset B$	A is a subset of B
$A \cup B$	the union of A and B
$A \cap B$	the intersection of A and B

Points and Sets in $\mathcal{R}^n$

What is $\mathcal{R}^n$?

$\mathcal{R}^n$ is the set $\{(x_1, x_2, ..., x_n) \mid x_i$ are real numbers$\}$. An element, or a point, of $\mathcal{R}^n$ is an n-tuple $(x_1, x_2, ..., x_n)$.

$\mathcal{R}^n$ is the Cartesian product of n $\mathcal{R}$s.

Example $\mathcal{R}^2 = \mathcal{R} \times \mathcal{R} = \{(x, y) \mid x,\ y$ belong to $\mathcal{R}\}$

Recall the following way to construct the product of two sets

Definition The Cartesian product of two sets X and Y is the set

$X \times Y = \{(x, y)|\ x$ belongs to X and y belongs to $Y\}$.

A rectangular region in $\mathcal{R}^n$ can be constructed as the Cartesian product of n intervals.

Example $[a, b] \times [c, d] = \{(x, y)|\ a \leq x \leq b\ \&\ c \leq y \leq d\}$ It is a closed subset of $\mathcal{R}^2$.

Rectangular (Cartesian) Coordinates

In Analytic Geometry, the location of a point in three-dimensional space can be determined by its coordinates with respect to a coordinate system.

Usually the system is rectangular; it means that the coordinate axes are perpendicular to each other.

Arrangement of the axes in 3D is according to the "Right-Hand Rule".

Important Open Sets

In the one dimensional case, Open neighborhoods

$$N_\delta(x_0) = \{x|\ |x - x_0| < \delta\}$$

and deleted open neighborhoods

$$\overline{N}_\delta(x_0) = \{x|\ 0 < |x - x_0| < \delta\}$$

are needed in defining 'limit'.

Definition We say that 'the number l is the limit of $f(x)$ as x approaches x_0' and write $\lim_{x \to x_0} f(x) = l$ if for any $\varepsilon > 0$ $\exists\, \delta > 0$ such that $x \in \overline{N}_\delta(x_0) \Rightarrow f(x) \in N_\varepsilon(l)$.

The corresponding open sets in two dimensions are:

(i) An open disk in $\mathcal{R}^2$, centered at (x_0, y_0) and with radius δ, is the set

$$D_\delta(x_0, y_0) = \{(x, y)|\ \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta\}.$$

(ii) A deleted open disk centered at (x_0, y_0) with radius δ is the set

$$\overline{D}_\delta(x_0, y_0) = \{(x, y)|\ 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta\}.$$

Elementary Functions

x^n, $\sin x$ and $\cos x$, e^x ($\exp x$) and $\ln x$

They are continuous and differentiable (infinitely many times) in their natural domains.

Vectors and Scalars

A vector is described by two things: a direction and a magnitude.

> Example What are supposed to be vectors?
> displacement, velocity, force,
> angular velocity, ...

A vector is denoted by $\overrightarrow{PQ}$ or $\vec{A}$, or $\bar{A}$. Its magnitude (or length) is denoted by $|\overrightarrow{PQ}|$ or $|\vec{A}|$, or simply A. P is the initial point; Q is the terminal point.

Abstract vectors discard the exact "locations" of the initial/terminal points. A vector can be concieved as a pointed stick which can be translated freely as long as the direction and length are unchanged.

A scalar is simply a real number; it is usually denoted by a lower-case alphabet.

Displacement vectors (or pointed sticks) can be used to illustrate the basic properties of general vectors.

Fundamental Properties & Operations of Vectors

1. Equality: $\vec{A} = \vec{B}$

   iff they have the same magnitude & direction

   locations of the initial points are irrelevant

2. Parallel vectors: $\vec{A} \parallel \vec{B}$

   iff they have the same (or opposite) direction

3. Negative vector: $-\vec{A}$

   has the same magnitude as $\vec{A}$, but opposite direction

4. Addition: $\vec{A} + \vec{B}$

   defined by the triangle rule

5. Subtraction: $\vec{A} - \vec{B}$

   defined as $= \vec{A} + (-\vec{B})$

   therefore, $\vec{A} - \vec{A} = \vec{0}$ (zero vector)

6. Multiplication by a scalar: $m\vec{A}$

   is a vector with magnitude $|m||\vec{A}|$ and direction the same or opposite to $\vec{A}$ according as m is > 0 or < 0. If $m = 0, m\vec{A} = \vec{0}$.

   Thus, for non-zero m and $\vec{A}$, $m\vec{A} \parallel \vec{A}$.

   Note Suppose that $\vec{A}$ and $\vec{B} \neq \vec{0}$. $\vec{A} \parallel \vec{B}$ iff $\exists\, m \neq 0$ such that $m\vec{A} = \vec{B}$.

   $\vec{0}$ is not parallel to any other vector.

The Algebra of Vectors

If $\vec{A}, \vec{B}, \vec{C}$ are vectors, and m, n are scalars, then:

1. $\vec{A} + \vec{B} = \vec{B} + \vec{A}$

   commutative law for vector addition

2. $\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$

   associative law for vector addition

3. $m(n\vec{A}) = (mn)\vec{A} = n(m\vec{A})$

   associate law for scalar multiplication

4. $(m + n)\vec{A} = m\vec{A} + n\vec{A}$

   distributive law for scalar multiplication

5. $m(\vec{A} + \vec{B}) = m\vec{A} + m\vec{B}$

   distributive law for scalar multiplication

Unit Vectors

— vectors with unit length

Normalizing a vector

If $|A| \neq 0$, $\hat{A} = \dfrac{\vec{A}}{|\vec{A}|}$ is the unit vector along the same direction as $\vec{A}$.

Rectangular Unit Vectors

$$\left.\begin{array}{l} \vec{i} \parallel \text{ x-axis} \\ \vec{j} \parallel \text{ y-axis} \\ \vec{k} \parallel \text{ z-axis} \end{array}\right\} \perp \text{ to each other}$$

Similar to the coordinate axes, $\vec{i}$, $\vec{j}$, $\vec{k}$ form a right-handed system.

They are called the basic unit vectors.

Correspondence Between Vectors and Points in $\mathcal{R}^n$

As long as the coordinate system is fixed, the correspondence between points in real space and the 3-tuples of coordinates are 1-1 and onto.

Recall that two vectors $\vec{A}$ and $\vec{B}$ are considered equal iff they have the same magnitude and direction, regardless of the initial point. Any vector $\vec{A}$ in 3 (or n) dimensions can then be represented with initial point placed at the origin O of a rectangular coordinate system. The terminal point will then fall on some point with label (A_1, A_2, A_3) in the coordinate system. The numbers A_1, A_2, A_3 are called the x, y, z components of $\vec{A}$; they describe $\vec{A}$ completely and uniquely.

The basic unit vectors can be specified as:

$$\vec{i} = (1, 0, 0), \quad \vec{j} = (0, 1, 0), \quad \vec{k} = (0, 0, 1).$$

$\vec{A}$ can be written as

$$\vec{A} = A_1\vec{i} + A_2\vec{j} + A_3\vec{k}.$$

where $A_1\vec{i}$, $A_2\vec{j}$, and $A_3\vec{k}$ are *vector components* of $\vec{A}$.

The magnitude of $\vec{A}$ can be computed as

$$|\vec{A}| = \sqrt{A_1^2 + A_2^2 + A_3^2}\,.$$

Vector Operations in Terms of Components

Vector addition

$$\vec{A} + \vec{B} = (A_1 + B_1)\vec{i} + (A_2 + B_2)\vec{j} + (A_3 + B_3)\vec{k}$$

Scalar multiplication

$$m\vec{A} = (mA_1)\vec{i} + (mA_2)\vec{j} + (mA_3)\vec{k}$$

Directed Line Segment

A 'directed line segment' is a straight line segment joining two points in space. The direction is from the initial point

$P = (P_1, P_2, P_3)$ to the terminal point $Q = (Q_1, Q_2, Q_3)$. The 'directed line segment *vector*' is

$$\overrightarrow{PQ} = (Q_1 - P_1)\vec{i} + (Q_2 - P_2)\vec{j} + (Q_3 - P_3)\vec{k}.$$

Example 1. Find $\overrightarrow{PQ}$ if $P = (1, 2, 3), Q = (4, 5, 6)$

2. Find a unit vector $\parallel$ $\overrightarrow{PQ}$

3. Check if $\vec{A} \parallel \vec{B}$

(a) $\vec{A} = \vec{i} + \vec{j} + \vec{k},\ \vec{B} = \sqrt{2}\vec{i} + \sqrt{2}\vec{j} + \sqrt{2}\vec{k}$

(b) $\vec{A} = \vec{i} + \vec{j} + \vec{k},\ \vec{B} = \sqrt{2}\vec{i} + \sqrt{2}\vec{j}$

Dot (or Scalar, or Inner) Product

Definition The dot product of two vectors $\vec{A}, \vec{B}$ is the real number

$\vec{A} \cdot \vec{B} = |A||B| \cos\theta \qquad 0 \le \theta \le \pi.$

θ is the angle between $\vec{A}$ and $\vec{B}$.

Projection

Let $\hat{B} = \vec{B}/\|\vec{B}\|$. The *scalar projection* of $\vec{A}$ onto $\vec{B}$ is

$$\vec{A} \cdot \hat{B} = (\vec{A} \cdot \vec{B})/\|\vec{B}\|.$$

It is the 'shadow' of $\vec{A}$ on a line L along $\vec{B}$.

The *vector projection* of $\vec{A}$ onto $\vec{B}$ is the vector

$$\text{proj}_{\vec{B}}\vec{A} = (\vec{A} \cdot \hat{B})\hat{B}.$$

It is the unit vector along $\vec{B}$ multiplied by the scalar projection of $\vec{A}$ onto $\vec{B}$.

<u>Basic properties of the dot product</u>

1. $\vec{A}\cdot \vec{B} = \vec{B}\cdot \vec{A}$ commutative
2. $\vec{A}\cdot (\vec{B} + \vec{C}) = \vec{A}\cdot \vec{B} + \vec{A}\cdot \vec{C}$ distributive

   (<u>pf</u>: use projections of $\vec{B}$, $\vec{C}$ onto $\vec{A}$)
3. $m(\vec{A}\cdot \vec{B}) = (m\vec{A})\cdot \vec{B} = \vec{A}\cdot (m\vec{B})$ m a scalar
4. $\vec{i}\cdot \vec{i} = \vec{j}\cdot \vec{j} = \vec{k}\cdot \vec{k} = 1$

   $\vec{i}\cdot \vec{j} = \vec{j}\cdot \vec{k} = \vec{k}\cdot \vec{i} = 0$
5. If $\vec{A} = A_1\vec{i} + A_2\vec{j} + A_3\vec{k}, B = B_1\vec{i} + B_2\vec{j} + B_3\vec{k}$, then $\vec{A}\cdot \vec{B} = A_1B_1 + A_2B_2 + A_3B_3$

   <u>Example</u> Find the dot product of $\vec{i} + \vec{j} + \vec{k}$ and $\vec{i} + 2\vec{j} - 3\vec{k}$.
6. If both $\vec{A}, \vec{B} \neq \vec{0}$, then $A \perp B$ iff $\vec{A}\cdot \vec{B} = 0$

   <u>Example</u> The components of $\vec{A}$ are scalar projections of $\vec{A}$ on the coordinate unit vectors.

think about it

<u>Example</u> Let $\vec{A} = \vec{i} + \vec{j} + \vec{k}$ and $\vec{B} = \vec{i} + \vec{j}$. Find the scalar projection of $\vec{A}$ on $\vec{B}$. What about $\vec{B}$ on $\vec{A}$? Plot the vectors and note their positions. What is θ? (35.3^o)

<u>Cross (or Vector) Product</u>

<u>Definition</u> Let $\vec{u}$ be a unit vector <u>perpendicular</u> to the plane containing $\vec{A}, \vec{B}$ and such that $\vec{A}, \vec{B}$, $\vec{u}$ form a right-handed system.

Then the cross product $\vec{A} \times \vec{B}$ can be defined as

$$\vec{A} \times \vec{B} = |A||B| \sin\theta \ \vec{u} \quad 0 \le \theta \le \pi$$

where θ is the angle between $\vec{A}$ and $\vec{B}$.

<u>Basic properties of the cross product</u>

1. $\vec{A} \times \vec{B} \perp$ both $\vec{A}$ and $\vec{B}$
2. $|\vec{A} \times \vec{B}| =$ area of a parallelogram with sides $\vec{A}$ and $\vec{B}$.
3. $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$ <u>not</u> commutative

   $\implies \vec{A} \times \vec{A} = \vec{0}$

4. $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$ distributive

use parallelograms in 2d

5. $m(\vec{A} \times \vec{B}) = (m\vec{A}) \times \vec{B} = \vec{A} \times (m\vec{B})$

   m a scalar

6. If both $\vec{A}, \vec{B} \neq 0$, then $\vec{A} \parallel \vec{B}$ iff $\vec{A} \times \vec{B} = \vec{0}$

7. $\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = 0$

   $\vec{i} \times \vec{j} = \vec{k}, \quad \vec{j} \times \vec{k} = \vec{i}, \quad \vec{k} \times \vec{i} = \vec{j}$

   How to remember cross products of the basic unit vectors ?

   Use cyclic ordering of $\vec{i}, \vec{j}, \vec{k}$

8. If $\vec{A} = A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$,

   $\vec{B} = B_1\vec{i} + B_2\vec{j} + B_3\vec{k}$, then

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

   You don't need to use the determinant if you do the cross products of the basic unit vectors directly.

Example Let $\bar{A} = 4\bar{i} - \bar{j} + 3\bar{k}$ and $\bar{B} = 2\bar{i} + \bar{j} - \bar{k}$. Find a vector perpendicular to $\bar{A}$ and $\bar{B}$.

$-2\vec{i} + 10\vec{j} + 6\vec{k}$

Triple Products

Scalar triple product

$$\vec{A}\cdot(\vec{B}\times\vec{C}) \;= \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

$|\vec{A}\cdot(\vec{B}\times\vec{C})|$ = volume of a parallelepiped with $\vec{A}$, $\vec{B}$, and $\vec{C}$ as edges

Vector triple product

$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$$

Example: $\vec{k}\times(\vec{j}\times\vec{k})$

do both sides of the formula