PART I: APPLICATION OF VECTORS

AND VECTOR-VALUED FUNCTIONS

Lines in Space

Vectors that connect any pair of points on a line l are $\parallel$.

Vector description of a line

Consider the locus of the terminal points of the vectors (first in 2D)

$$t\vec{A} \qquad -\infty < t < \infty$$

It is a straight line. This line can be displaced to describe any straight line $\parallel$ to $\vec{A}$.

The general vector expression for a line in space is:

$$\vec{r} = \vec{r}_0 + t\vec{A} \qquad -\infty < t < \infty$$

where $\vec{r}_0$ is an arbitrary point on the line and $\vec{A}$ is a vector $\parallel$ to the line.

Example Find a vector equation of the line that contains $(-1, 3, 0)$ and is parallel to $2\bar{i} - 3\bar{j} - \bar{k}$

Example Find a vector equation of the line that passes through the two points: $(1, 2, 3)$ and $(4, 5, 6)$.

Note $\vec{r}(t)$ gives the location of a point on the line. It is not parallel to the line. $\vec{A}$ is. Namely, if $\vec{P}$ and $\vec{Q}$ are any two different points on the line, then $\vec{PQ} \parallel \vec{A}$.

Parametric equations

$$
\begin{aligned}
x &= x_0 + tA_1 \\
y &= y_0 + tA_2 \\
z &= z_0 + tA_3
\end{aligned}
$$

Example Find a set of parametric equations for the line of the previous example.

Analytic geometry (AG) description of a line

$$
\begin{cases}
a_1 x + b_1 y + c_1 z = d_1 \\
a_2 x + b_2 y + c_2 z = d_2
\end{cases}
$$

Example Find an AG description of the line in the previous example.

Degrees of Freedom / Dimensions

$\equiv$ number of free parameters used to describe the geometrical object

Example Let $\vec{A}$ be a vector in space, the object $t\vec{A}$ $(-\infty < t < \infty)$ has one free parameter, therefore the object has dimension 1.

Example If $\vec{A}$ and $\vec{B}$ are not parallel, and both $\neq \vec{0}$, what is the object

$t\vec{A} + s\vec{B} \quad -\infty < t < \infty \quad -\infty < s < \infty$?

Planes in 3D Space

Basic geometrical facts

(i) Two different intersecting lines l_1, l_2 (directions given by $\vec{A}$, $\vec{B}$ respectively) define a plane through them.

(ii) There is a unique line l_3 that passes through the point of intersection and is $\perp$ to both lines. A vector $\vec{N} \parallel l_3$ defines a normal to the plane that contains l_1, l_2. $(\vec{N} \cdot \vec{A} = \vec{N} \cdot \vec{B} = 0)$

(iii) There is only one plane that contains a given point and is $\perp$ to a given nonzero vector.

(i) can be viewed as the definition of a plane; (ii) & (iii) come from the three-dimensional nature of space.

Vector description of a plane

A plane through two intersecting lines l_1 and l_2 can be generated as:

$$\vec{r} = \vec{r}_0 + t\vec{A} + s\vec{B} \qquad -\infty < t < \infty,\ -\infty < s < \infty$$

where $\vec{r}_0$ is the point of intersection of l_1, l_2, and $\vec{A}, \vec{B}$ are vectors along l_1, l_2 respectively.

A normal is perpendicular to all directed line segments on the plane as $\vec{N} \cdot (t\vec{A} + s\vec{B}) = 0$.

Recall (ii) above for the definition of a normal. A vector is a normal to a plane iff it is perpendicular to all vectors associated with the directed line segments on the plane.

Parametric description of a plane:

Decompose the vector equation into components, as (linear) functions of the parameters.

$$\begin{aligned} x &= x_0 + tA_1 + sB_1 \\ y &= y_0 + tA_2 + sB_2 \\ z &= z_0 + tA_3 + sB_3 \end{aligned}$$

This approach is not used much for planes, but very useful for general surfaces. The coordinates of a point on a surface can be specified as:

$$\begin{aligned} x &= f_1(t,s), \\ y &= f_2(t,s), \\ z &= f_3(t,s). \end{aligned}$$

These are two-variable functions which make components of a function

$$f : \mathcal{R}^2 \to \mathcal{R}^3.$$

AG description of a plane:

Let P be a fixed point on the plane, Q be any point on the plane, and $\vec{N}$ be a normal. Then

$$\vec{N} \cdot \vec{PQ} = 0$$

Writing $\vec{P} = (x_0, y_0, z_0)$, $\vec{Q} = (x, y, z)$, and $\vec{N} = (a, b, c)$, one has

$$\begin{aligned} &(a,b,c) \cdot [(x,y,z) - (x_0,y_0,z_0)] = \\ &a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \end{aligned}$$

The resultant equation is of the form

$$ax + by + cz = d.$$

A normal to the plane can be found easily as

$$\vec{N} = a\vec{i} + b\vec{j} + c\vec{k}.$$

<u>Example:</u> Find an equation of the plane that contains the point $(-2, 4, 5)$ and has normal vector $7\vec{i} - 6\vec{k}$

<u>Example:</u> Find a unit normal to the plane $x + y + z = 1$.

<u>Example:</u> Find the plane through the three points $(0, 0, 1)$, $(1, 0, 1)$, $(0, 1, 1)$.

<u>Quadric Surfaces</u>

<u>Definition:</u> A quadric surface is a surface which contains points that satisfies the 2nd degree polynomial equation (AG description)

$$Ax^2+By^2+Cz^2+Dxy+Eyz+Fzx+Gx+Hy+Iz+J = 0$$

where A, ..., J are constants.

Quadric surfaces fall into nine major classes. Examples of each are provided as following ($a,\ b,\ c\ > 0$):

1. Ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$
2. Hyperboloid of one sheet $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$
3. Hyperboloid of two sheets $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$
4. Elliptic double cone $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 0$
5. Elliptic paraboloid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z}{c}$
6. Hyperbolic paraboloid $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = \dfrac{z}{c}$
7. Elliptic cylinder $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$
8. Hyperbolic cylinder $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$
9. Parabolic cylinder $x^2 = \dfrac{z}{c}$

—— Problem Set 1 ——