PART III: VECTOR FIELDS

[*We are getting less and less rigorous!*]

Definition A vector field $\vec{F}$ consists of two parts: a collection D of points in space, called the domain, and a rule, which assigns to each point (x, y, z) in D one and only one vector $\vec{F}(x, y, z)$.

Physical examples: electric field, gravity field, fluid flows (water, air)

Plot examples including source & sink, vortex.

The Gradient of a Scaler Function Creates a Vector Field

$$\text{grad } f = \vec{\nabla} f = f_x \vec{i} + f_y \vec{j} + f_z \vec{k}$$

Definition If a vector field $\vec{F}$ is equal to $\vec{\nabla} f$ for some differentiable function f (of several variables), then $\vec{F}$ is called a conservative vector field, and f is a potential function of $\vec{F}$.

Note In physics, $-f$ is used as the potential of a force field.

Example Electric field of a point charge q.

$$\vec{E} = \frac{q}{r^2}\hat{r}$$

$$f = \frac{-q}{r} = \frac{-q}{(x^2 + y^2 + z^2)^{1/2}}$$

$\hat{r}$ denotes the unit vector pointing along $\vec{r}$.

Divergence and Curl

Definition Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ be a vector field such that $\partial F_1/\partial x, \partial F_2/\partial y, \partial F_3/\partial z$ exist.

Then the divergence of $\vec{F}$, denoted $div\vec{F}$ or $\vec{\nabla}\cdot\vec{F}$, is the scalar function defined by

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

If $\vec{\nabla}\cdot\vec{F} = 0$, then $\vec{F}$ is said to be divergence-free.

Source and Sink

Let $\vec{F}(x, y, z)$ be a vector field in space (example: a flow field), then

A point (x, y, z) is a source if $\vec{\nabla} \cdot \vec{F} > 0$

A point (x, y, z) is a sink if $\vec{\nabla} \cdot \vec{F} < 0$

Example Consider the divergence of the field

$$\vec{F} = ax\, \vec{i}$$

for cases with $a >, =,$ and < 0.

Example Consider the field of a uniformly charged sphere

$$\vec{E} = \begin{cases} r\hat{r} & \text{for} \quad r \leq a \\ \dfrac{a^3}{r^2}\hat{r} & \text{for} \quad r > a \end{cases}$$

$$\vec{\nabla} \cdot \vec{E} = \begin{cases} 3 & \text{for} \quad r < a \\ \text{undefined} & \text{for} \quad r = a \\ 0 & \text{for} \quad r > a \end{cases}$$

Note The divergence of a radial inverse square field is zero everywhere, except at the center.

Definition Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ be a vector field such that the first derivatives of F_1, F_2, F_3 all exist. Then the curl of F, denoted *curl* $\vec{F}$ or $\vec{\nabla} \times \vec{F}$, is defined by

$$\vec{\nabla} \times \vec{F}(x, y, z) = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \times \vec{F}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}.$$

Example Let $\vec{F}(x, y, z) = xz\vec{i} + xy^2z\vec{j} - e^{2y}\vec{k}$. Find $\vec{\nabla} \times \vec{F}$

$[-(2e^{2y} + xy^2)\vec{i} + x\vec{j} + y^2z\vec{k}]$

Definition If $\vec{\nabla} \times \vec{F} = 0$, $\vec{F}$ is said to be irrotational.

Example Consider the flow

$$\vec{V} = +\Omega r\hat{\theta}$$

where Ω is a constant (angular velocity).

a. Plot the flow.

b. Find $\vec{\nabla} \times \vec{V}$ and $\vec{\nabla} \cdot \vec{V}$.

Note that since $\hat{\theta} = -\sin\theta\vec{i} + \cos\theta\vec{j}$, $\vec{V} = \Omega(-y\vec{i} + x\vec{j})$.

Formulas Involving $\vec{\nabla}$

1. $\vec{\nabla}(f+g) = \vec{\nabla} f + \vec{\nabla} g$

   $\vec{\nabla}\cdot(\vec{F}+\vec{G}) = \vec{\nabla}\cdot\vec{F} + \vec{\nabla}\cdot\vec{G}$

   $\vec{\nabla}\times(\vec{F}+\vec{G}) = \vec{\nabla}\times\vec{F} + \vec{\nabla}\times\vec{G}$

2. $\vec{\nabla}(fg) = f\nabla g + (\nabla f)g$

   $\vec{\nabla}\cdot(f\vec{G}) = f\vec{\nabla}\cdot\vec{G} + (\vec{\nabla} f)\cdot\vec{G}$

   $\vec{\nabla}\times(f\vec{G}) = f\vec{\nabla}\times\vec{G} + (\vec{\nabla} f)\times G$

   Special cases: f = a constant c

3. (no need to memorize the following)

   $$\vec{\nabla}(\vec{F}\cdot\vec{G}) = (\vec{F}\cdot\vec{\nabla})\vec{G} + (\vec{G}\cdot\vec{\nabla})\vec{F} + \vec{F}\times(\vec{\nabla}\times\vec{G}) + \vec{G}\times(\vec{\nabla}\times\vec{F})$$

   $$\vec{\nabla}\cdot(\vec{F}\times\vec{G}) = \vec{G}\cdot(\vec{\nabla}\times\vec{F}) - \vec{F}\cdot(\vec{\nabla}\times\vec{G})$$

   $$\vec{\nabla}\times(\vec{F}\times\vec{G}) = (\vec{G}\cdot\vec{\nabla})\vec{F} - (\vec{F}\cdot\vec{\nabla})\vec{G} + \vec{F}(\vec{\nabla}\cdot\vec{G}) - \vec{G}(\vec{\nabla}\cdot\vec{F})$$

3. Theorem: $\vec{\nabla}\times(\vec{\nabla} f) = \vec{0}$ $\quad$ (*curl grad* $= 0$)

   $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) = 0$ $\quad$ (*div curl* $= 0$)

Example For $r \neq 0$ the curl of the electric field $\vec{E} = \dfrac{q}{r^2}\hat{r}$ is $\vec{0}$.

Note $\vec{\nabla} \times (\vec{\nabla} \times \vec{F}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{F}) - \nabla^2 \vec{F}$ is generally not $\vec{0}$.

Example Show that the field of the example in the previous subsection is the curl of another vector field:

$$\vec{V} = \Omega r \hat{\theta} = \Omega(-y\vec{i} + x\vec{j}) = \Omega \vec{\nabla} \times \left(-\frac{r^2}{2} \vec{k} \right),$$

$$\text{So that } \vec{\nabla} \times \vec{\nabla} \times \left(-\frac{r^2}{2} \Omega \vec{k} \right) = 2\Omega \vec{k}$$

— Problem Set 10 —