Line Integrals

Let C be a smooth, oriented (directed) curve in space. $\vec{F}$ is a continuous vector field. Approximate C by a collection of small line segments (directed) $\{\triangle\vec{r}_i\}$. Consider the sum $\sum\limits_i \vec{F}(\xi_i)\cdot\triangle\vec{r}_i$ where ξ_i is a point on the line segment $\triangle\vec{r}_i$ (the concept of work), then take the limit $|\triangle r_i| \to 0$

$$\sum_i \vec{F}(\xi_i)\cdot\triangle\vec{r}_i \quad\to\quad \int_C \vec{F}\cdot d\vec{r}$$

This is called the line integral of $\vec{F}$ over C.

If C is not smooth but is piecewise smooth, composed of smooth curves $C_1, C_2, \cdots C_n$, then

$$\int_C \vec{F}\cdot d\vec{r} = \sum_{i=1}^{n}\int_{C_i}\vec{F}\cdot d\vec{r}.$$

Also, with $-C$ defined to be the curve having the same points but opposite orientation of C,

$$\int_{-C}\vec{F}\cdot d\vec{r} = -\int_C \vec{F}\cdot d\vec{r}.$$

As $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ and $d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$, another form to write a line integral is

$$\int_C \vec{F} \cdot d\vec{r} = \int_C F_1 dx + F_2 dy + F_3 dz.$$

<u>Evaluation of Line Integrals</u>

Let $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ be a parameterization of C with domain $[a, b]$, and assume that the <u>parameterization induces</u> the given <u>orientation</u> (direction) of C. Then (the formula for evaluation)

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F} \circ \vec{r}(t) \cdot \frac{d\vec{r}}{dt} dt\,.$$

<u>Example</u> A particle moves upward along the circular helix C, parameterized by $\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j} + t\vec{k}$ for $0 \leq t \leq 2\pi$ under a force given by $\vec{F}(x, y, z) = -zy\vec{i} + zx\vec{j} + xy\vec{k}$. Find the work done on the particle by the force (i.e. the line integral). $2\pi^2$

<u>Example</u> Assume that the particle in the previous example moves under the same force and with the same initial and terminal points,

but along the line segment C_2 parameterized by

$$\vec{r}(t) = \vec{i} + t\vec{k} \quad \text{for} \quad 0 \le t \le 2\pi.$$

Find the work done on the particle by the force. 0

Note that $\int_{C_1} \neq \int_{C_2}$

The Fundamental Theorem of Line Integrals

Theorem Let C be an oriented curve with initial point (x_0, y_0, z_0) and terminal point (x_1, y_1, z_1). Let f be a function of three variables that is differentiable at every point on C, and assume that $\vec{\nabla} f$ is continuous on C. Then

$$\int_C \vec{\nabla} f \cdot d\vec{r} = f(x_1, y_1, z_1) - f(x_0, y_0, z_0)$$

pf: Use Chain Rule.

Example Let C be the straight line segment from $(0, 2, 0)$ to $(1, 0, 0)$ and $\vec{F}$ be the electric field of a point charge q at the origin. Find the work done by $\vec{F}$ on a unit point charge that traverses C. $-q/2$

Path Independence

Definition If a vector field $\vec{F}$ has the property that $\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$ for any two oriented curves having the same initial and terminal points, its line integrals are called path independent.

Clearly, $\vec{F} = \vec{\nabla} f \implies \int_C \vec{F}\cdot \vec{dr}$ path independent.

(Fundamental Theorem of Line Integrals)

On the other hand, if the line integrals of $\vec{F}$ are path independent, then a potential function of $\vec{F}$ can be found as $f(x,y,z) = \int_C \vec{F}\cdot \vec{dr}$ where C is an arbitrary curve connecting the origin (or any fixed reference point) to (x,y,z).

Therefore, $\int_C \vec{F}\cdot \vec{dr}$ path independent $\implies \vec{F} = \vec{\nabla} f$.

pf: $\vec{r}(s)=(s,y_0,z_0)$, $s\in[x_0,x]$, $\partial_x f(x_0,y_0,z_0)=(d/dx)\int_{x_0}^{x} F_1(s,y_0,z_0)ds$

Example Let $\vec{F} = xy^2\vec{i} + x^2y\vec{j}$.

a. Evaluate the line integrals $\int\limits_{C_1} \vec{F}\cdot d\vec{r}$ and $\int\limits_{C_2} \vec{F}\cdot d\vec{r}$ where C_1 consists of the line segments connecting $(0,0)$ to $(x_0,0)$ and $(x_0,0)$ to (x_0,y_0), and C_2 consists of the line segments connecting $(0,0)$ to $(0,y_0)$ and $(0,y_0)$ to (x_0,y_0). $x_0^2y_0^2/2$

b. Find a potential function for $\vec{F}$.

Important Properties of a Conservative Field

Theorem The following statements are equivalent:

1. $\vec{F} = \vec{\nabla} f$ for some function f, i.e. conservative.
2. $\displaystyle\int\limits_C \vec{F}\cdot d\vec{r}$ is independent of path.
3. $\displaystyle\int\limits_C \vec{F}\cdot d\vec{r} = 0$ for every closed loop C.

If the domain of $\vec{F}$ is a region with no holes, then also

4. $\vec{\nabla} \times \vec{F} = 0$.

Green's Theorem

Theorem Let R be a simple region in the xy plane with a piecewise smooth boundary C oriented counterclockwise. Let F_1 and F_2 be functions of two variables having continuous partial derivatives on R. Then

$$\int_C F_1(x,y)dx + F_2(x,y)dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dA$$

Note Writing $\vec{F}$ as $F_1\vec{i} + F_2\vec{j}$ and considering the situation in 3 dimensions, this equation can be expressed as

$$\int_C \vec{F}\cdot d\vec{r} = \iint_R (\vec{\nabla}\times\vec{F}\ )\cdot\vec{k}dA$$

Pf: It is sufficient to show that

$$\int_C F_2(x,y)dy = \int_C F_2\vec{j}\cdot d\vec{r} = \iint_R \frac{\partial F_2}{\partial x}dA$$

and

$$\int_C F_1(x,y)dx = \int_C F_1\vec{i}\cdot d\vec{r} = -\iint_R \frac{\partial F_1}{\partial y}dA.$$

Note that the separation is done through writing $\vec{F} = \vec{F_1} + \vec{F_2}$ where $\vec{F_1} = F_1\vec{i}$ and $\vec{F_2} = F_2\vec{j}$.

$$-\iint\limits_R \frac{\partial F_1}{\partial y} dA = -\int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial F_1}{\partial y} dy dx$$

$$= \int_a^b [F_1(x, g_1(x)) - F_1(x, g_2(x))] dx$$

The first term is the line integral $\int_{C_1} F_1 \vec{i} \cdot d\vec{r}$ on the curve C_1 parameterized by $\vec{r_1}(x) = x\vec{i} + g_1(x)\vec{j}$. The second term is $-\int_{C_2} F_1 \vec{i} \cdot d\vec{r}$ where C_2 is the curve parameterized and oriented by $\vec{r_2}(t) = x\vec{i} + g_2(x)\vec{j}$, $x \in [a, b]$.

<u>Example</u> Find $\int\limits_C -x^2 y dx + x^3 dy$ where C is the circle $x^2 + y^2 = 4$, oriented conterclockwise.

16π *(both ways)*

—— Problem Set 11 ——