## Oriented Surfaces and Flux Integrals

Let $\Sigma$ be a surface that has a tangent plane at each of its nonboundary points. At such a point on the surface two unit normal vectors exist, and they have opposite directions.

If it is possible to select one normal at each nonboundary point in such a way that the chosen normal varies continuously on the whole of $\Sigma$, then the surface $\Sigma$ is said to be orientable, or two-sided, and the selection of the normal gives an orientation to $\Sigma$ and thus makes $\Sigma$ an oriented surface. In such case, there are two possible orientations. Some surfaces are not orientable. For example, the Möbus band; it is one-sided.

## Induced Orientation

If $\Sigma$ is an oriented surface bounded by a curve $C$, then the orientation of $\Sigma$ induces an orientation for $C$, based on the Right-Hand-Rule.

## Flux Integrals

Let $\vec{F}$ be a vector field and $\hat{n}$ be the unit normal to the oriented surface $\Sigma$, the flux integral over $\Sigma$ is

$$\iint_{\Sigma} \vec{F} \cdot \hat{n} \, dS.$$

This integral gives the net flux through $\Sigma$. The field strength (i.e. $|\vec{F}|$) can be measured as the amount of *flux per unit area* perpendicular to the local direction of $\vec{F}$.

When $\Sigma$ is the graph of a function $f$ with continuous partials on a region $R$ in the $xy$ plane that is composed of vertically or horizontally simple regions, and its orientation is chosen to be directed upward (i.e. the $\vec{k}$ component of the unit normal is positive), then

$$\iint\limits_{\Sigma} \vec{F} \cdot \hat{n} dS = \iint\limits_{R} [-F_1 f_x - F_2 f_y + F_3] dA$$

for $\vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}$.

Note Another way to remember this is

$$\iint\limits_{\Sigma} \vec{F} \cdot \hat{n} dS = \iint\limits_{R} \vec{F} \cdot \vec{N} dA$$

where $\vec{N} = -f_x \vec{i} + -f_y \vec{j} + \vec{k}$ is the upward normal to $\Sigma$ with the $z$ component equal to 1.

Example Suppose $\Sigma$ is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the $xy$ plane and is oriented by the unit normal directed upward. Assume that the velocity of a fluid is $\vec{v} = x\vec{i} + y\vec{j} + 2z\vec{k}$. Determine the flow rate (volume/time) through $\Sigma$.

$2\pi$

Flux integrals can be defined for a surface $\Sigma$ composed of several oriented surfaces $\Sigma_1, \Sigma_2, \cdots \Sigma_n$, as

$$\iint\limits_{\Sigma} \vec{F} \cdot \hat{n} dS = \iint\limits_{\Sigma_1} \vec{F} \cdot \hat{n} dS + \cdots\cdots + \iint\limits_{\Sigma_n} \vec{F} \cdot \hat{n} dS$$

Example Let $\Sigma$ be the unit sphere $x^2 + y^2 + z^2 = 1$, oriented with the unit normal directed outward, and let $F(x, y, z) = z\vec{k}$. Find $\iint\limits_{\Sigma} \vec{F} \cdot \hat{n} dS$.

$4\pi/3$

## The Divergence Theorem

Definition A solid region $D$ is called a simple solid region if $D$ is the solid region between the graphs of two functions $f(x,y)$ and $g(x,y)$ on a simple region $R$ in the $xy$ plane and if $D$ has the corresponding properties with respect to the $xz$ plane and the $yz$ plane.

Theorem Let $D$ be a simple solid region whose boundary surface $\Sigma$ is oriented by the normal $\hat{n}$ directed outward from $D$, and let $\vec{F}$ be a vector field whose component functions have continuous partial derivatives on $D$. Then

$$\iint_{\Sigma} \vec{F}\cdot \hat{n} dS = \iiint_{D} \vec{\nabla}\cdot \vec{F} dV.$$

pf: Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$, then

$$\iint_{\Sigma} F_1\vec{i}\cdot \hat{n} dS = \iiint_{D} \frac{\partial F_1}{\partial x} dV$$

$$\iint_{\Sigma} F_2\vec{j}\cdot \hat{n} dS = \iiint_{D} \frac{\partial F_2}{\partial y} dV$$

$$\iint_{\Sigma} F_3\vec{k}\cdot \hat{n} dS = \iiint_{D} \frac{\partial F_3}{\partial z} dV$$

Note The interpretation of $\iint_\Sigma \vec{F}\cdot\hat{n}\,dS$ is "net outward flux through the boundary of $D$".

Example Let $D$ be the region bounded by the $xy$ plane and the hemisphere $x^2+y^2+z^2=4$ with $z\geq 0$, and let $\vec{F}(x,y,z) = 3x^4\vec{i} + 4xy^3\vec{j} + 4xz^3\vec{k}$.

Evaluate $\displaystyle\iint_\Sigma \vec{F}\cdot\hat{n}dS$, where $\Sigma$ is the boundary of $D$.

0

Note If the domain of integration $R$ is symmetry with respect to certain thing and the integrand is anti-symmetric w.r.t. the same thing, then the integral is 0.

For example, if $f(-x,y,z) = -f(x,y,z)$, the function $f$ is anti-symmetric w.r.t. the $yz$-plane.

The theorem can be applied to a 'not so simple' solid region by considering the division of the region into a number of simple solid regions.

Example Let $\vec{F} = \frac{q}{r^2}\hat{r}$.

Show that for any suface $\Sigma$ enclosing the origin, $\iint\limits_{\Sigma} \vec{F} \cdot \hat{n} dS = 4\pi q$.

Stokes' Theorem

Theorem Let $\Sigma$ be an oriented surface with normal $\hat{n}$ (unit vector) and finite surface area. Assume that $\Sigma$ is bounded by a closed, piecewise smooth curve $C$ whose orientation is induced by $\Sigma$. Let $\vec{F}$ be a continuous vector field defined on $\Sigma$, and assume that the component functions of $\vec{F}$ have continuous partial derivatives at each nonboundary point of $\Sigma$. Then

$$\int_C \vec{F}\cdot d\vec{r} = \iint_\Sigma (\vec{\nabla} \times \vec{F})\cdot \hat{n} dS$$

Example Let $C$ be the intersection of the paraboloid $z = x^2+y^2$ and the plane $z = y$, and give $C$ a counterclockwise direction as viewed from the positive $z$ axis. Evaluate

$$\int_C 2xydx + x^2dy + z^2dz.$$

What about

$$\int_C (2xy - y)dx + x^2dy + z^2dz \quad ?$$

0; $\pi/4$

Example Verify Stokes' Theorem for $\vec{F} = 3y\vec{i} - xz\vec{j} + yz^2\vec{k}$ where $\Sigma$ is the surface of the paraboloid $2z = x^2 + y^2$ bounded by $z = 2$, and $C$ is its boundary.

$\vec{\nabla} \times \vec{F} = (z^2 + x)\vec{i} - (z+3)\vec{k}$

$-20\pi$

—— Problem Set 12 ——

Proof of Stokes' theorem

$$(\vec{\nabla} \times \vec{F}) \cdot \vec{N} = (\partial_y F_3 - \partial_z F_2)(-f_x) + (\partial_z F_1 - \partial_x F_3)(-f_y) + (\partial_x F_2 - \partial_y F_1)$$

$$= (\partial_x F_2 + f_x \partial_z F_2 + f_y \partial_x F_3) - (\partial_y F_1 + f_y \partial_z F_1 + f_x \partial_y F_3)$$

$$= \partial_x (F_2(x, y, f(x, y)) + f_y F_3(x, y, f(x, y)) - \partial_y (F_1(x, y, f(x, y)) + f_x F_3(x, y, f(x, y))$$

Now apply Green's theorem on the $xy$-plane as following.

$$\iint_R \left( \frac{\partial}{\partial x}(F_2 + f_y F_3) - \frac{\partial}{\partial y}(F_1 + f_x F_3) \right) dA$$

$$= \oint\limits_{\partial R} (F_1 + f_x F_3)dx + (F_2 + f_y F_3)dy$$

$\vec{r}(t)=(x(t),y(t),z(t)) \ \ where \ \ z(t)=f(x(t),y(t))$

$So \ \ that \ \ \vec{r}'(t)=(x'(t),y'(t),f_x x'(t)+f_y y'(t))$

$and \ \ \vec{F}\cdot\vec{r}'(t)=F_1 x'+F_2 y'+(F_3 f_x x'+F_3 f_y y')$

$$= \oint\limits_{\partial\Sigma} F_1 dx + F_2 dy + F_3 dz.$$

Therefore,

$$\iint\limits_{\Sigma} (\vec{\nabla} \times \vec{F}) \cdot \hat{n} dS = \iint\limits_{R} (\vec{\nabla} \times \vec{F}) \cdot \vec{N} dA = \oint\limits_{\partial\Sigma} \vec{F} \cdot d\vec{r}.$$