Vector-Valued Functions

Definition: A vector-valued function consists of two parts: a domain, which is a subset of $\mathcal{R}$, and a rule, which assigns to each number in the domain one and only one vector.

The *rule* is usually given as a *formula*. The formula of the function can be expressed as

$$\vec{r}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$$

where t is in the common domain of f_1, f_2, f_3 (single-variable functions called the component functions).

The *range* (or *image*) of a vector-valued function is a curve in space. $\{\vec{p} \mid \vec{p} = \vec{r}(t)\ t \in domain\}$

Example: $\vec{A}(t) = (1+2t)\vec{i} + 3t\vec{j} + \frac{1}{t}\vec{k}$

$\vec{B}(t) = \sqrt{1+2t}\ \vec{i} + 4t\vec{k}$

$\vec{C}(t) = \sin t\ \vec{i} + t\vec{k} \qquad t \in [0, 2\pi]$

domain of $\vec{A}$ *excludes the point* $t=0$

domain of $\vec{B} = \{t \mid t \geq -\frac{1}{2}\}$

plot $\vec{C}(t)$

Limits of Vector-Valued Functions

Definition Let $\vec{A}$ be a vector-valued function defined at each point in some open interval containing t_0, except possibly at t_0 itself. A vector $\vec{L}$ is the limit of $\vec{A}(t)$ as t approaches t_0 if for every $\epsilon > 0 \quad \exists$ a $\delta > 0$ such that

$$|\vec{A}(t) - \vec{L}| < \varepsilon \quad \text{whenever} \quad 0 < |t - t_0| < \delta.$$

We write $\lim_{t \to t_0} \vec{A}(t) = \vec{L}$.

Detailed discussion of 'limit' will be postponed to the "several-variable functions" section.

Theorem Let $\vec{A}(t) = A_1(t)\vec{i} + A_2(t)\vec{j} + A_3(t)\vec{k}$. Then $\vec{A}$ has a limit at t_0 if and only if A_1, A_2, and A_3 have limits at t_0. In that case

$$\lim_{t \to t_0} \vec{A}(t) = [\lim_{t \to t_0} A_1(t)]\vec{i} + [\lim_{t \to t_0} A_2(t)]\vec{j} + [\lim_{t \to t_0} A_3(t)]\vec{k}.$$

Example Find $\lim_{t \to 0}(2\cos t\ \vec{i} + \frac{\sin t}{t}\vec{j} + t^2\vec{k})$.

Useful Rules

Assuming that $\lim_{t\to t_0} \vec{A}(t)$, $\lim_{t\to t_0} \vec{B}(t)$, $\lim_{t\to t_0} f(t)$ exist, then

1. $\lim_{t\to t_0} (\vec{A} \pm \vec{B})(t) = \lim_{t\to t_0} \vec{A}(t) \pm \lim_{t\to t_0} \vec{B}(t)$
2. $\lim_{t\to t_0} [f\vec{A}(t)] = \lim_{t\to t_0} f(t) \lim_{t\to t_0} \vec{A}(t)$
3. $\lim_{t\to t_0} (\vec{A}\cdot \vec{B})(t) = \lim_{t\to t_0} \vec{A}(t)\cdot \lim_{t\to t_0} \vec{B}(t)$
4. $\lim_{t\to t_0} (\vec{A} \times \vec{B})(t) = \lim_{t\to t_0} \vec{A}(t) \times \lim_{t\to t_0} \vec{B}(t)$

Continuity

Definition A vector-valued function $\vec{A}$ is continuous at a point t_0 in its domain if

$$\lim_{t\to t_0} \vec{A}(t) = \vec{A}(t_0).$$

Theorem: A vector-valued function $\vec{A}$ is continuous at t_0 if and only if each of its component functions is continuous at t_0.

Example: $\vec{A}(t) = 2\cos t\ \vec{i} + \dfrac{\sin t}{t}\vec{j} + t^2\vec{k}$ is discontinuous at $t = 0$.

Theorem If $\vec{A}(t)$ is continuous at t_0 and $\lim_{s\to s_0} g(s) = t_0$, then

$$\lim_{s\to s_0}(\vec{A}(g(s)) = \vec{A}(\lim_{s\to s_0} g(s)) = \vec{A}(t_0).$$

Derivatives

Definition Let t_0 be a number in the domain of a vector-valued function $\vec{A}$. If

$$\lim_{t\to t_0} \frac{\vec{A}(t) - \vec{A}(t_0)}{t - t_0}$$

exists, we call this limit the derivative of $\vec{A}$ at t_0 and write it as $\vec{A}'(t_0)$, or $\frac{d}{dt}\vec{A}(t_0)$, or $\dot{A}(t_0)$. We say that $\vec{A}$ is differentiable at t_0.

Theorem $\vec{A}'(t_0) = A_1'(t_0)\vec{i} + A_2'(t_0)\vec{j} + A_3'(t_0)\vec{k}$

Example Find the derivative of
$\vec{A}(t) = e^t \cos t\vec{i} - ln(t)\vec{j}$

Useful Formulas for Differentiation

Let $\vec{A}, \vec{B}$, and f be differentiable at t_0, and let g be differentiable at s_0 with $g(s_0) = t_0$. Then

1. $(\vec{A} \pm \vec{B})'(t_0) = \vec{A}'(t_0) \pm \vec{B}'(t_0)$

2. $(f\vec{A})'(t_0) = f'(t_0)\vec{A}(t_0) + f(t_0)\vec{A}'(t_0)$
3. $(\vec{A}\cdot \vec{B})'(t_0) = \vec{A}'(t_0)\cdot \vec{B}(t_0) + \vec{A}(t_0)\cdot \vec{B}'(t_0)$
4. $(\vec{A} \times \vec{B})'(t_0) = \vec{A}'(t_0) \times \vec{B}(t_0) + \vec{A}(t_0) \times \vec{B}'(t_0)$
5. $\frac{d}{ds}\vec{A}(g)(s_0) = \vec{A}'(g(s_0))g'(s_0) = \vec{A}'(t_0)g'(s_0)$

Curves in Space

We will generally use the symbol C to denote a curve. If the range of a *continuous* vector-valued function $\vec{r}$ is C, we say that $\vec{r}$ is the parameterization of C, and C is parameterized by $\vec{r}$.

A curve C is closed if it has a parameterization whose domain is a closed interval $[a, b]$ such that $\vec{r}(a) = \vec{r}(b)$.

A vector-valued function $\vec{r}(t)$ defined on an interval I is smooth if $\vec{r}$ has a continuous derivative on I and $\vec{r}'(t) \neq \vec{0}$ for each interior point t. A curve C is smooth if it has a smooth parameterization.

Example Consider the curve parameterized by

$$\vec{r}(t) = t^3\vec{i} + t^2\vec{j}.$$

plot with the substitution $s=t^3$

Higher-Order Derivatives

$$\vec{A}''(t_0) = \lim_{t \to t_0} \frac{\vec{A}'(t) - \vec{A}'(t_0)}{t - t_0}$$

Interpretation of Derivatives

Tangent vector to a point on a curve

Velocity and acceleration

Suppose $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ is position and t is time. Then

Velocity: $\vec{v}(t) = \frac{d}{dt}\vec{r}(t) = \dot{\vec{r}} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$

speed: $|\vec{v}(t)| = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}$

Acceleration: $\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = \ddot{\vec{r}}$

Example Show that the velocity vector of a particle moving on a sphere $(radius = r_0)$ is always perpendicular to the radial vector.

Example Show that the projection of the acceleration vector of a particle moving on a sphere on the radial direction is $-\dfrac{|\dot{\vec{r}}|^2}{|\vec{r}|}\hat{r}$.

Arc Length of a Curve

If C is the image of a smooth vector-valued function $\vec{r}(t)$, then its arc length L from $t = a$ to $t = b$ is

$$L = \int_a^b \left\| \frac{d\vec{r}}{dt} \right\| dt.$$

Example Find the arc length of that portion of the circular helix

$$x = \cos t, \quad y = \sin t, \quad z = t$$

from $t = 0$ to $t = \pi$.

—— Problem Set 2 ——