# PART II: MULTI-VARIABLE CALCULUS

## Functions of Several Variables

Definition A function of several variables consists of two parts: a domain, which is a set of points in the plane or in space, and a rule, which assigns to each member of the domain one and only one real number.

Example Area of a rectangle *note domain limitation*

Example $f(x, y) = 2x^2y^2 + 4x^2 - 7y^2 + 2y + \sqrt{3}$ is an example of a polynomial formula (terms are of the form $cx^my^n$).

Note A formula (like the $f(x, y)$ above) is not a function; it is only a rule of assignment.

The natural domain of a formula for n-independent variables consists of n-tuples that can be properly evaluated by the formula (to produce real numbers).

Example Find the natural domain of

$$f(x, y, z) = \sqrt{xyz^2}.$$

## Generating New Functions from Known Functions

### Algebraic combination of functions

$$(f \pm g)(x, y) = f(x, y) \pm g(x, y)$$
$$(f \cdot g)(x, y) = f(x, y)g(x, y)$$
$$\left(\frac{f}{g}\right)(x, y) = \frac{f(x, y)}{g(x, y)} \text{ for } g(x, y) \neq 0$$

Note The domain of the new function is (a subset of) the intersection of those of the original functions.

### Composite function

If $g$ is a single variable function and $f$ is a multivariable function (say of two variables),

$$(g \circ f)(x, y) = g(f(x, y))$$

is the formula of $g \circ f$.

There are two conditions that restrict the domain of $g \circ f$, $\mathcal{D}_{g \circ f}$: (i) $(x, y) \in \mathcal{D}_f$, (ii) $f(x, y) \in \mathcal{D}_g$. Therefore,

$$\mathcal{D}_{g \circ f} = \mathcal{D}_f \cap \{(x, y) |\ f(x, y) \in \mathcal{D}_g\}.$$

Example Consider the domain of $g \circ f$ where $g(x) = \sqrt{x}$ and $f(x, y) = y$.

*some* $f(x,y)$ *fall outside domain of* $g$ $(-\infty,\infty)\times[0,\infty)$

Example Consider the domain of $g \circ f$ where $g(x) = x^2$ and $f(x, y) = \sqrt{xy}$.

*natural domain of* $g(f(x,y))$ *bigger than domain of* $f$

## Graph and Level Curves of a two-variable function

Definition The graph of a function $f$ of two variables is the collection of points $(x, y, f(x, y))$ for which $(x, y)$ is in the domain of $f$

Definition The set of points $(x, y)$ in the $xy$ plane such that $f(x, y) = c$ is a level curve of $f$.

Note Different level curves (i.e. different $c$) do not intersect.

Example Let $f(x, y) = \sin y$. Sketch the graph.

Example Let

$$f(x, y) = \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}.$$

Plot level curves $f = 0,\ 1/2,\ 1,\ 2$, and sketch the graph of $f$.

(an example of quadric surfaces)

*watch domain*

## Level surfaces of a three-variable function

Definition The set of points $(x, y, z)$ in space such that $f(x, y, z) = c$ is a level surface of $f$.

Example Let $g(x, y, z) = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - z^2$.
Sketch the level surfaces $g(x, y, z) = c$ for $c = -1,\ 0,\ 1,\ 2$.

## Limits and Continuity

Definition In the $xy$-plane, the open disk centered at $(x_0, y_0)$ with a radius $\delta$ is the set

$D_\delta(x_0, y_0) = \{(x, y)|\ |(x, y) - (x_0, y_0)| < \delta\}$

The deleted open disk $\overline{D}_\delta(x_0, y_0)$ is

$\{(x, y)|\ 0 < |(x, y) - (x_0, y_0)| < \delta\}$.

Note In 3D, we use the open ball $B_\delta(x_0, y_0, z_0)$ & deleted open ball $\overline{B}_\delta(x_0, y_0, z_0)$.

If $A$ is a subset of the domain of $f$, the image of $A$ can be written as

$f(A) = \{y|\ y = f(x), x \in A\}$.

Definition Let $f(x, y)$ be defined throughout an open disk centered at $(x_0, y_0)$, except possibly at

$(x_0, y_0)$ itself, and let $l$ be a number. Then $l$ is the limit of $f$ at $(x_0, y_0)$ if for every $\varepsilon > 0 \ \exists$ a $\delta > 0$ such that if $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$, then $|f(x, y) - l| < \varepsilon$.

In this case we write

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = l,$$

and say that $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists.

Another way to write the requirement is:
$\forall\varepsilon > 0,\ \exists\delta > 0$ such that $f(\overline{D}_\delta(x_0, y_0)) \subset N_\varepsilon(l)$.

Example Show that

$$\lim_{(x,y)\to(x_0,y_0)} x = x_0 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} y = y_0$$

Limit Formulas

If $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ and $\displaystyle\lim_{(x,y)\to(x_0,y_0)} g(x,y)$ exist, then

1. $$\lim_{(x,y)\to(x_0,y_0)} (f \pm g)(x,y) = \lim_{(x,y)\to(x_0,y_0)} f(x,y) \pm \lim_{(x,y)\to(x_0,y_0)} g(x,y)$$

2. $$\lim_{(x,y)\to(x_0,y_0)} (fg)(x,y) = \lim_{(x,y)\to(x_0,y_0)} f(x,y) \lim_{(x,y)\to(x_0,y_0)} g(x,y)$$

3. $$\lim_{(x,y)\to(x_0,y_0)} \left(\frac{f}{g}\right)(x,y) = \frac{\lim_{(x,y)\to(x_0,y_0)} f(x,y)}{\lim_{(x,y)\to(x_0,y_0)} g(x,y)}$$

   if $\displaystyle\lim_{(x,y)\to(x_0,y_0)} g(x,y) \neq 0$

Example Show that

a. $$\lim_{(x,y)\to(0,1)} \frac{x^3+y^3}{x^2+y^2} = 1$$

b. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = 0$

$use\ \ |a+b| \leq |a| + |b|$

$Note:\ \lim_{(x,y)\to(0,0)} \equiv \lim_{\sqrt{x^2+y^2}\to 0}$

Theorem If a limit exists, it is a unique number.

$pf\ \ by\ \ contradiction$

Theorem Suppose that $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y) = l.$

Let $\vec{r} : (a,b) \to \mathcal{R}^2$ be a continuous vector-valued function so that $\displaystyle\lim_{t\to b^-} \vec{r}(t) = (x_0, y_0)$ but $\vec{r}(t) \neq (x_0, y_0)$ for any $t \in (a,b)$, then $\displaystyle\lim_{t\to b^-} (f \circ \vec{r})(t) = l.$

These theorems are useful for showing the non-existence of limits with: $\displaystyle\lim_{along\ path\ 1} \neq \lim_{along\ path\ 2}.$

Example Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.

Example Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 y}{2x^6+y^2}.$

$y=mx,\ \ y=x^3$

Example What about $\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x + y}$?

Note the subtlety of the definition of limit.

Substitution Rule (for composite functions)

If $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$ and $g$ is a single-variable function continuous at $L$, then

$$\lim_{(x,y)\to(x_0,y_0)} g(f(x, y)) = g(L).$$

In other words,

$$\lim_{(x,y)\to(x_0,y_0)} g(f(x, y)) = g(\lim_{(x,y)\to(x_0,y_0)} f(x, y)).$$

*Note that $f$ need not be defined at $(x_0,y_0)$*

Example Find $\lim_{(x,y)\to(0,0)} \ln(\frac{\sin\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}})$

Example Let

$$g(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

and $f(x, y) = xy$, show that

$\lim_{(x,y)\to(0,0)} g \circ f(x, y)$ does not exist.

## Continuity

Definition Suppose $f$ is a function of two variables defined throughout an open disk about $(x_0, y_0)$. Then $f$ is continuous at $(x_0, y_0)$ iff

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0).$$

Definition If a function $f$ is continuous at each point in its domain, then it is said to be a continuous function.

Note A composite of continuous functions is continuous.

Example

$$F(x,y) = \sin\frac{xy}{1+x^2+y^2}$$

is a continuous function.

—— Problem Set 3 ——