Limit and Continuity on a Set (general case)

Open and close sets

Let R be a set in the plane. Then for each point P in the plane, one and only one of the following possibilities holds:

1. $\exists$ an open disk centered at P and contained totally in R. In this case P is an interior point of R.
2. $\exists$ an open disk centered at P and containing no points of R. In this case P is an exterior point of R.
3. Every open disk centered at P contains a point in R and a point outside of R. In this case P is a boundary point of R.

Definition The collection of boundary points of R is the boundary of R.

Definition If a set R contains its boundary, then R is closed.

Definition If a set R contains only interior points, then R is open.

Example The open disk $D_\varepsilon(x_0, y_0) =$
$\{(x,y) | \sqrt{(x-x_0)^2 + (y-y_0)^2} < \varepsilon\}$
is open.

Example The set $\{(x,y) | \ \ |x| < 1, |y| \leq 1\}$ is neither open nor close.

Limit and continuity at a boundary point

Definition Let (x_0, y_0) be on the boundary of R. A number L is the limit of f restricted to R at (x_0, y_0) if for every $\varepsilon > 0, \exists$ a number $\delta > 0$ such that if $(x, y) \in R$ and $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$, then $|f(x, y) - L| < \varepsilon$. Then, we write

$$\lim_{(x,y)_R \to (x_0,y_0)} f(x, y) = L.$$

Example Let R be the natural domain of

$$f(x, y) = \frac{x^2 - y^2}{x + y}.$$

Consider $\lim\limits_{(x,y)_R \to (0,0)} f(x, y)$.

Continuity at a boundary point (x_0, y_0) can then be defined by the condition

$$\lim_{(x,y)_R \to (x_0,y_0)} f(x, y) = f(x_0, y_0).$$

Continuity on a set (general situation)

Definition If f is continuous at every interior point of R and $\lim\limits_{(x,y)_R \to (x_0,y_0)} f(x, y) = f(x_0, y_0)$

for every boundary point (x_0, y_0) in R, we say that f is continuous on R.

Theorem Let R be a close, bounded set in the plane, and let f be continuous on R. Then f has both a maximum and a minimum on R.

Definition A set is 'bounded' if there exists a number M so that the distances of all its points to the origin are less than M.

Example Consider $f(x, y) = xy$ on $[0, 1] \times [0, 1] - \{(1, 1)\}$

pf by contradiction. $x<1$ *or* $y<1$

Partial Derivatives

Use graph, mention the tangent lines (plane) to motivate

Definition Let f be a function of two variables, and let (x_0, y_0) be in the domain of f. The partial derivative of f with respect to (w.r.t.) x at (x_0, y_0) is defined by

$$f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

provided that this limit exists.

The notations $\dfrac{\partial}{\partial x} f$, $\partial_x f$ are also in use.

The partial derivative of f w.r.t. y at (x_0, y_0) can be similarly defined.

Example Let $z = \sin(xy^2)$. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

The tangent line to the curve $(x, y_0, f(x, y_0))$ through the point (x_0, y_0, z_0) where $z_0 = f(x_0, y_0)$ is described by the vector equation $(x, y, z) = (x_0, y_0, z_0) + (x - x_0)(1, 0, f_x)$. Similarly the tangent line to the curve $(x_0, y, f(x_0, y))$ can be found. The plane that contains these two lines is given by $-f_x(x - x_0) - f_y(y - y_0) + (z - f(x_0, y_0)) = 0$.

Formulas

$$(f \pm g)_x = f_x \pm g_x$$

$$(fg)_x = f_x g + f g_x$$

$$(f/g)_x = \frac{f_x g - f g_x}{g^2} \qquad \text{for } g \neq 0$$

Similarly for $(\ )_y$

Higher-Order Partial Derivatives

2nd order

$$(f_x)_x \qquad f_{xx} \qquad \frac{\partial^2 f}{\partial x^2}$$

$$(f_x)_y \qquad f_{xy} \qquad \frac{\partial^2 f}{\partial y \partial x} \qquad \text{mixed partials}$$

$$(f_y)_x \qquad f_{yx} \qquad \frac{\partial^2 f}{\partial x \partial y} \qquad \text{mixed partials}$$

$$(f_y)_y \qquad f_{yy} \qquad \frac{\partial^2 f}{\partial y^2}$$

Example Let $f(x, y) = \sin(xy^2)$. Find all 2nd order partial derivatives of f (illustrate that $f_{xy} = f_{yx}$).

$f_{xy} = -2xy^3 \sin(xy^2) + 2y\cos(xy^2)$

Theorem Let f be a function of two variables, and assume that f_{xy} and f_{yx} are continuous at (x_0, y_0). Then

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0).$$

Since polynomials, trigonometric functions, exponential and logarithmic functions are continuously differentiable

everywhere (in their domains), mixed partials of their composites can have the orderings switched under most circumstances.

Though a necessary requirement, the existence of f_x and f_y does not guarantee differentiability. It does not even guarantee continuity.

Example Consider

$$f(x,y) = \begin{cases} 0 & \text{on the } x \text{ and } y \text{ axes} \\ 1 & \text{otherwise} \end{cases}$$

Differentiability

Definition Let $\triangle x = x - x_0$ and $\triangle y = y - y_0$. If f is such that $\triangle f = f(x,y) - f(x_0,y_0)$ can be expressed in the form $\triangle f = f_x \triangle x + f_y \triangle y + \epsilon_1 \triangle x + \epsilon_2 \triangle y$ where $\epsilon_1, \epsilon_2 \to 0$ as $\triangle x, \triangle y \to 0$, we call f differentiable at (x_0, y_0).

Consider the plane though the two first-partial tangents.

Note This definition is equivalent to the regular definition in the single-variable situation. *prove first*

Example Consider $f(x,y) = xy$. At $(0,0)$, $f = 0$, $f_x = 0$ and $f_y = 0$. As $\Delta x = x$ and $\Delta y = y$, one can pick, for example, $\epsilon_1 = y/2$ and $\epsilon_2 = x/2$.

Theorem If f_x and f_y are continuous at (x_0, y_0), f is differentiable at the point.

Theorem differentiability $\Rightarrow$ continuity

Definition A function f is called differentiable on a region R if it is differentiable at each point of R.

Tangent Plane

If $f(x, y)$ is differentiable, a tangent plane to its graph at the point $(x_0, y_0, f(x_0, y_0))$ can be defined. The equation

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

describes the tangent plane that passes through the point. A normal to this plane is $(f_x(x_0, y_0), f_y(x_0, y_0), -1)$.

Example Let $f(x, y) = \sqrt{1 - x^2 - y^2}$, find the tangent plane to the graph of this function at $(0, 0, 1)$.

normal $\|$ *position vector.*

Tangent Plane Approximation

When f is differentiable at (x_0, y_0), the value of f at a point (x, y) near (x_0, y_0) can be approximated by

$$f(x, y) - f(x_0, y_0) \approx f_x(x_0, y_0)\triangle x + f_y(x_0, y_0)\triangle y.$$

i.e. $f(x, y) \approx z$ while (x, y, z) is a point on the tangent plane through $(x_0, y_0, f(x_0, y_0))$. The exact value of $f(x, y)$ is not necessarily z, but it can be closely approximated by z (the error terms are $\epsilon_1 \triangle x + \epsilon_2 \triangle y$).

(graphical illustration of the tangent plane approximation).

Example A rectangular cardboard box has outer dimensions 30, 30, and 20 cm. If the cardboard is 3 mm thick, estimate the volume of cardboard.

$\Delta x = 6mm; \ \Delta V \approx 1260cc$

Differentials

As $\triangle x, \triangle y$ become very small, the approximation above can be written in the form

$$df = f_x(x, y)dx + f_y(x, y)dy.$$

It is called the total differential of f at (x, y). dx and dy are called the differentials of x and y, respectively.

—— Problem Set 4 ——