Differentiation of Composite Functions

Chain rules: Assuming that f, r_1, r_2, h_1, h_2 are *differentiable*:

1. Let $z = f(x,y), x = r_1(t)$ and $y = r_2(t)$. Then $z = f(r_1(t), r_2(t))$, and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \quad \text{(diff. along a curve)}$$

pf use defn of differentiability

$\vec{r}(t) = r_1(t)\vec{i} + r_2(t)\vec{j}$ is a vector-valued function that traces a curve on the (x,y) plane.

Note $z(t) = (f \circ \vec{r})(t)$ also stands for the composite function, and dx/dt stands for $dr_1(t)/dt$.

2. Let $z = f(x,y), x = h_1(u,v)$ and $y = h_2(u,v)$. Then $z = f(h_1(u,v), h_2(u,v))$, and

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

(use directed graphs)

Example Let $z = x^2 e^y, x = \sin t$, and $y = t^3$. Find $\dfrac{dz}{dt}$.

<u>Example</u> Let $z = xln(y), x = u^2 + v^2$, and $y = u^2 - v^2$. Find $\partial z/\partial u$ and $\partial z/\partial v$.

<u>Example</u> (implicit differentiation)

a. Given $y + \sin(yx^2) = 1$, find dy/dx.

$-2xy \cos x^2 y/(1+x^2 \cos x^2 y)$

b. Assuming that $f(x, y) = 0$ defines a differentiable function $y = g(x)$ of x, so that $f(x, g(x)) = 0$. Find g' in terms of the partial derivatives of f.

<u>Directional Derivatives</u>

– differentiation along a direction different from the x-, y-axes.

<u>Definition</u> Let f be a function defined on a set containing an open disk centered at (x_0, y_0), and let $\hat{u} = u_1\vec{i} + u_2\vec{j}$ be a <u>unit vector</u>. Then the <u>directional derivative</u> of f at

(x_0, y_0) in the direction of $\hat{u}$, denoted

$D_{\hat{u}} f(x_0, y_0)$, is defined by

$$D_{\hat{u}} f(x_0, y_0) = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

provided that this limit exists.

<u>Note</u> s is a distance parameter along the direction of $\hat{u}$. $|s|$ measures the distance to (x_0, y_0).

<u>Theorem</u> Let f be differentiable at (x_0, y_0). Then f has a directional derivative at (x_0, y_0) in <u>every direction</u>.

Moreover, if $\hat{u} = u_1\vec{\imath} + u_2\vec{\jmath}$,

$$D_{\hat{u}}f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2.$$

Pf: Let $g_1(s) = x_0 + su_1$ & $g_2(s) = y_0 + su_2$ and use the Chain Rule.

<u>Example</u> Let $f(x, y) = xy^2$ and $\vec{u} = \vec{\imath} - 2\vec{\jmath}$. Find the directional derivative of f at $(-3, 1)$ in the direction of $\vec{u}$. $13/\sqrt{5}$

<u>The Gradient</u>

<u>Definition</u> Grad $f(x_0, y_0)$

$$= \vec{\nabla} f(x_0, y_0) = f_x(x_0, y_0)\vec{\imath} + f_y(x_0, y_0)\vec{\jmath}$$

<u>Theorem</u> a. $D_{\hat{u}}f(x_0, y_0) = \hat{u} \cdot \vec{\nabla} f(x_0, y_0)$

b. $D_{\hat{u}}f(x_0, y_0) = ||\vec{\nabla} f(x_0, y_0)|| \cos\phi$

(ϕ is the angle between $\vec{u}$ and $\vec{\nabla} f$)

Note $D_{\hat{u}} f(x_0, y_0)$ is a scalar while $\vec{\nabla} f(x_0, y_0)$ is a vector.

The largest value of $D_{\hat{u}} f$ is $||\vec{\nabla} f||$, and this value is obtained when $\hat{u}$ points in the direction of $\vec{\nabla} f$.

Example Let $f(x,y) = 6 - 3x^2 - y^2$. Determine the directions in which f increases/decreases most rapidly at $(1,2)$ and find the maximal value of the directional derivative.

$\vec{\nabla} f(1,2) = (-6,-4)$, $||\vec{\nabla} f(1,2)|| = \sqrt{52}$

The Gradient as a Normal Vector

Let C be a level curve $f(x,y) = c$ of a function f. Let (x_0, y_0) be a point on C, and assume that f is differentiable at (x_0, y_0). If C is smooth and $\vec{\nabla} f(x_0, y_0) \neq \vec{0}$, then $\vec{\nabla} f(x_0, y_0)$ is normal to C at (x_0, y_0).

Pf.: Let $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$ be a smooth parameterization of C, then

$$\frac{d}{dt} f(x(t), y(t)) = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = \frac{d\vec{r}}{dt} \cdot \vec{\nabla} f = 0.$$

Note The first version of the Chain Rule can be written as

$$\frac{d(f \circ \vec{r})(t)}{dt} = \vec{\nabla} f \cdot \frac{d\vec{r}}{dt}$$

Example Find a unit vector $\perp$ to the curve $x^2 - xy + 3y^2 = 5$ at $(1, -1)$. $(1/\sqrt{58})(3, -7)$

Theorem Let S be a level surface of a function f and (x_0, y_0, z_0) is a point on S. If f is differentiable at (x_0, y_0, z_0) and $\vec{\bigtriangledown} f(x_0, y_0, z_0) \neq \vec{0}$, then $\vec{\bigtriangledown} f(x_0, y_0, z_0)$ is $\perp$ to the tangent vector at (x_0, y_0, z_0) of any smooth curve lying on S and passing through this point. Therefore, $\vec{\bigtriangledown} f(x_0, y_0, z_0)$ is a normal to the tangent plane of S at (x_0, y_0, z_0).

Example Find an equation of the plane tangent to the sphere $x^2 + y^2 + z^2 = 4$ at $(-1, 1, \sqrt{2})$.

Now suppose that f is a function of two variables that is differentiable at (x_0, y_0). In order to obtain an equation of the plane tangent to the graph of f at (x_0, y_0), one can think of the graph of f as the level surface $g(x, y, z) = 0$ where $g = f(x, y) - z$. Then

$$\vec{\bigtriangledown} g(x_0, y_0, z_0) = f_x(x_0, y_0)\vec{i} + f_y(x_0, y_0)\vec{j} - \vec{k}$$

provides a normal to the tangent plane of the graph at (x_0, y_0).

Example Find an equation for the tangent plane to the graph of

$$f(x,y) = 6 - 3x^2 - y^2 \ \text{ at } \ (1,2,-1).$$

Taylor Series

Can $f(x,y) = f(x_0 + \Delta x, y_0 + \Delta y)$ be expressed in terms of $f(x_0,y_0)$ and the partial derivatives of f at (x_0,y_0)?

Let $(x_0,y_0) + s\hat{u}$ be the parameterization of a line that passes through (x_0,y_0) and (x,y), with $\hat{u}$ being the unit vector

$$u_1\vec{i} + u_2\vec{j} \equiv \frac{1}{\sqrt{\Delta x^2 + \Delta y^2}}(\Delta x\vec{i} + \Delta y\vec{j}).$$

The parameter s is the distance from (x_0,y_0) to a point on the line.

The function $f(x_0 + su_1, y_0 + su_2)$ can be considered as a single-variable function of s. From the discussion of directional derivative, we know that

$$\frac{d}{ds}f = D_u f = (u_1\frac{\partial}{\partial x} + u_2\frac{\partial}{\partial y})f$$

in which $D_{\hat{u}}$ is written as a *differential operator.*

With $\hat{u}$ fixed, $D_{\hat{u}}f$ can be considered as a function of (x,y). Suppose that f is sufficiently differentiable, then

$$\frac{d^2}{ds^2}f = D_u^2 f, ...$$

Considered as a function of s, the Taylor series of $f(x_0 + su_1, y_0 + su_2)$ can be written as

$$= f(x_0, y_0) + sD_u f + \frac{1}{2!}s^2 D_u^2 f + \cdots + \frac{1}{n!}s^n D_u^n f + \cdots$$

By choosing $s = \sqrt{\Delta x^2 + \Delta y^2}$, one obtains

$$sD_u f = (\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y})f,$$

$$s^2 D_u^2 f = (\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y})^2 f,$$

$$\cdots\cdots,$$

and

$$f(x_0 + \Delta x, y_0 + \Delta y) = f(x_0, y_0) + (\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y})f$$

$$+ \frac{1}{2!}(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y})^2 f + \cdots + \frac{1}{n!}(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y})^n f + \cdots$$

assuming that all relevant derivatives exist.

Useful for determining max/min.

—— Problem Set 5 ——